AMERICAN

Journal of Mathematics

EDITED BY

FRANK MORLEY

WITH THE COOPERATION OF

A. COHEN, CHARLOTTE A. SCOTT

AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

Πραγμάτων ἔλεγχος οὐ βλεπομένων

VOLUME XL

BALTIMORE: THE JOHNS HOPKINS PRESS

LEMCKE & BUECHNER; *New York.* E. STEIGER & CO., *New York.* A. HERMANN, *Paris.*
G. E. STECHERT & CO., *New York.* ARTHUR F. BIRD, *London.* MAYER & MÜLLER, *Berlin.*
WILLIAM WESLEY & SON, *London.*

1918

The Lord Baltimore Press
BALTIMORE, MD., U. S. A.

INDEX

Flat-Sphere Geometry.

SECOND PAPER.

BY JOHN EIESLAND.

INTRODUCTION.

In a paper published in AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV,* I have extended Lie's Line-Sphere Geometry in 3-space to an odd-dimensional space in which a special self-dual element takes the place of the line in 4-space. Conforming to the notation of this memoir, to which we shall refer in the following pages by the letter A, we denote the odd-dimensional space by S_{n-1} (n-even). The following theorem was proved (A, p. 226):

There exist in $\overline{\overline{S}}_{n-1}\infty^{\frac{n+1\cdot n+2}{2}}$ *contact-transformations which transform the* ∞^n *flat spreads*

$$x_i=ay_i+b_i,\quad z=\Sigma c_iy_i+d,\quad \{q_i=-ap_i+c_i\},\qquad i=1, 2, \ldots, \frac{n-2}{2}. \qquad (1)$$

into the ∞^n *spheres in* S_{n-1}. *These transformations are obtained by superposing the inverse of the generalized Lie transformation L on the contact-transformations that leave the* ∞^n *flats* (1) *invariant. These transformations have the characteristic functions:*

$$\left.\begin{array}{l} 1,\ p_i,\ y_i,\ x_i,\ q_i,\ x_ip_i+y_iq_i,\ x_ky_i-x_iy_k,\ q_kp_i-p_kq_i,\ p_ix_k+q_iy_k, \\ x_ip_k+y_iq_k,\ \ zy_k-y_k\Sigma q_jy_j-x_k\Sigma p_jy_j,\ \ zp_k-p_k\Sigma q_jy_j+q_k\Sigma p_jy_j,\ \ \Sigma q_jy_j, \\ \Sigma x_jq_j,\ \Sigma p_jy_j,\ 2z-\Sigma(x_jp_j+y_jq_j),\ zx_k-y_k\Sigma x_jq_j-x_k\Sigma x_jp_j,\ zq_k+p_k\Sigma x_jq_j \\ -q_k\Sigma x_jq_j,\ z^2-z\Sigma q_jy_j-z\Sigma p_jx_j+\Sigma x_jp_j\cdot\Sigma qyy-\Sigma x_jq_j\cdot\Sigma y_jp_j, \\ \hfill i, k=1, 2, \ldots, \frac{n-2}{2},\ i\neq k. \end{array}\right\} \qquad (2)$$

* "On a Flat Spread-Sphere Geometry in Odd-Dimensional Space," pp. 201–228.

The inverse of L is:

$$\left.\begin{aligned}
x_i &= -(X_{2i-1}+iX_{2i}) - \frac{P_{2i-1}+iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \quad y_i = -\frac{P_{2i-1}+iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \\
p_i &= \frac{P_{2i-1}-iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \quad q_i = -(X_{2i-1}-iX_{2i}) - \frac{P_{2i-1}-iP_{2i}}{1\pm\sqrt{1+\Sigma P_i^2}}, \\
z &= \frac{\Sigma(P_{2i-1}+iP_{2i})(X_{2i-1}-iX_{2i})}{1\pm\sqrt{1+\Sigma P_i^2}} - X_{n-1}, \quad i=1,2,\ldots,\frac{n-2}{2}.
\end{aligned}\right\} \quad (3)$$

The same transformations $L^{-1}G_{n+1\cdot\frac{n+2}{2}}$ transform the asymptotic curves on an $n-2$-spread M_{n-2} in $\overline{\overline{S}}_{n-1}$ into the lines of curvature on the transform of M_{n-2} in S_{n-1}. The differential equations of the asymptotic curves on M_{n-2} are

$$dx_i dp_{\frac{n-2}{2}} + dy_i dq_{\frac{n-2}{2}} = 0, \quad dq_k dp_{\frac{n-2}{2}} - dp_k dq_{\frac{n-2}{2}} = 0, \quad \begin{Bmatrix} i=1,2,\ldots,\frac{n-2}{2} \\ k=1,2,\ldots,\frac{n-2}{2} \end{Bmatrix}, \quad (4)$$

and those of the lines of curvature on the transform are

$$(dX_i + P_i X_{n-1})\, dP_{n-2} - (dX_{n-2} + P_{n-2} dX_{n-1})\, dP_i = 0, \quad i=1,2,\ldots,n-3.$$

I

§ 1. The lines that are here denoted as "asymptotic" are from the standpoint of the flat-sphere geometry in S_{n-1} the analogues of asymptotic lines on a surface in 3-space, inasmuch as through any point on the surface there pass $n-2$ such lines, and to these correspond by the generalized Lie transformation (3) the lines of curvature on the transform.*

The lines of curvature on a spread M_{n-2} in S_{n-1} are not necessarily *coordinate lines* of curvature in the Darboux sense. In fact, as Darboux has shown,† an $n-2$-spread has coordinate lines of curvature if and only if an $n-1$-tuple orthogonal system exists of which the given M_{n-2} is one of the $n-1$ mutually orthogonal surfaces. If a surface has coordinate lines of curvature

* *I. e., in general;* in the case of special surfaces we may have "bands" of curvature or curvature = spreads, in which case the transform has "bands" of asymptotic curves. Such cases we shall meet with in the present paper.

† Darboux, "Leçons sur les systèmes orthogonaux et les coordinées curvilignes," pp. 133–137 and 176–182. See also a note in *Comptes Rendus*, Vol. CXXVIII, pp. 284–285, entitled "Sur les systèmes orthogonaux," by A. Pellet.

it may be shown that the coordinates X_i and the direction cosines of the normal c_i must satisfy the generalized Olinde Rodrigue formulae

$$\frac{\partial X_i}{\partial \rho_k}+R_k\frac{\partial c_i}{\partial \rho_k}=0, \qquad \begin{array}{l} i=1, 2, \ldots, n-1, \\ k=1, 2, \ldots, n-2, \end{array}$$

that is, on a surface having coordinate lines of curvature the functions $\rho_1, \ldots, \rho_{n-2}$ are such that on a line of curvature ρ_k the function ρ_k only varies while the others remain constant. For 3-space this is always true, while for $n>3$ this condition imposes restrictions on the surface as an individual; while every surface in n-space has lines of curvature, not every surface is a member of an n-tuple orthogonal system of surfaces.

We shall say that a spread M_{n-2} in $\overline{\overline{S}}_{n-1}$ has *coordinate asymptotic lines* whenever its transform by the transformation (3) has coordinate lines of curvature.

Through the researches of Darboux and others we now know a great many orthogonal systems in n-space, and therefore also an indefinite number of surfaces having coordinate lines of curvature. If in an odd space we transform these by (3) we obtain surfaces on which the corresponding lines are coordinate asymptotic lines, but since in general, to real surface-elements in S_{n-1} correspond imaginary elements in $\overline{\overline{S}}_{n-1}$, this method is not always convenient and sometimes even impractical, since the eliminations to be performed are rather complicated if the surface in S_{n-1} is to be obtained in parametric or Cartesian form. The lines of curvature being real on a real surface, the corresponding asymptotic lines will in general be imaginary if the transform is a real surface.

§ 2. We shall represent the tangent flat F_{n-2} to an n—2-spread in S_{n-1} by the following equation:

$$(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\ldots+i(\alpha_{\frac{n-2}{2}}-\beta_{\frac{n-2}{2}})X_{n-2} \\ +(1-\Sigma\alpha_i\beta_i)X_{n-1}+F=0. \quad (5)$$

the envelope of which is the spread M_{n-2} (A, p. 208):

$$\left.\begin{array}{l} X_1=\dfrac{(\alpha_1+\beta_2)}{2}X_{n-1}-\dfrac{1}{2}(F'_{\alpha_1}+F'_{\beta_1}), \\ X_2=i\dfrac{(\alpha_1-\beta_1)}{2}X_{n-1}+\dfrac{i}{2}(F'_{\alpha_1}-F'_{\beta_1}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-1}=\dfrac{\Sigma\alpha_i F'_{\alpha_i}+\Sigma\beta_i F'_{\beta_i}-F}{1+\Sigma\alpha_i\beta_i}. \end{array}\right\} \quad (6)$$

The equations (5) may therefore be considered as the tangential equation of the spread. The equations of the lines of curvature are (A, p. 204):

$$\frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\beta_i}}{dF'_{\beta_{\frac{n-2}{2}}}},\quad \frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\alpha_k}}{dF'_{\beta_{\frac{n-2}{2}}}},\quad \begin{array}{l} i=1,2,\ldots,\frac{n-4}{2}, \\ k=1,2,\ldots,\frac{n-2}{2}, \end{array} \tag{7}$$

which may be put in the form, writing $\overline{p}_i$ for F'_{α_i} and $\overline{q}_i$ for F'_{β_i},

$$d\alpha_i d\overline{q}_{\frac{n-2}{2}}-d\alpha_{\frac{n-2}{2}}d\overline{q}_i=0,\quad d\beta_k d\overline{q}_{\frac{n-2}{2}}-d\alpha_{\frac{n-2}{2}}d\overline{p}_k=0. \tag{8}$$

If $\rho_1, \ldots, \rho_{n-2}$ are coordinate lines of curvature we must have

$$\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\overline{q}_{\frac{n-2}{2}}}{\partial\rho_k}-\frac{\partial\alpha_{\frac{n-2}{2}}}{\partial\rho_k}\frac{\partial\overline{q}_i}{\partial\rho_k}=0,\qquad \frac{\partial\beta_i}{\partial\rho_k}\frac{\partial\overline{q}_{\frac{n-2}{2}}}{\partial\rho_k}-\frac{\partial\alpha_{\frac{n-2}{2}}}{\partial\rho_k}\frac{\partial\overline{p}_i}{\partial\rho_k}=0, \tag{9}$$

which may be written

$$\frac{\partial\overline{q}_i}{\partial\rho_k}=\lambda_k\frac{\partial\alpha_i}{\partial\rho_k},\quad \frac{\partial\overline{p}_i}{\partial\rho_k}=\lambda_k\frac{\partial\beta_i}{\partial\rho_k},\qquad \begin{pmatrix} i=1,2,\ldots,\frac{n-2}{2} \\ k=1,2,\ldots,\frac{n-2}{2} \end{pmatrix}, \tag{10}$$

to which must also be added the $n-2$ conditions

$$\frac{\partial F}{\partial\rho_k}=\sum\left[\overline{p}_i\frac{\partial\alpha_i}{\partial\rho_k}+\overline{q}_i\frac{\partial\beta_i}{\partial\rho_k}\right]. \tag{11}$$

Expressing the conditions for the integrability of (10) we obtain the following system of $\frac{(n-2)(n-3)}{2}$ partial differential equations which must be satisfied by the α's and β's:

$$(\lambda_k-\lambda_{k'})\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\theta}{\partial\rho_{k'}}=0. \tag{12}$$

In the same way, expressing the condition for the integrability of (11) we are led to the system of $\frac{n-2\cdot n-3}{2}$ equations:

$$\sum\left[\frac{\partial\overline{p}_i}{\partial\rho_{k'}}\frac{\partial\alpha_i}{\partial\rho_k}+\frac{\partial\overline{q}_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}-\frac{\partial p_i}{\partial\rho_k}\frac{\partial\alpha_i}{\partial\rho_{k'}}-\frac{\partial\overline{q}_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right]=0,$$

which by means of (10) may be reduced to the following:

$$(\lambda_k-\lambda_{k'})\sum\left[\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right]=0. \tag{13}$$

Assuming that $\lambda_k \neq \lambda_{k'}$ for all values of k and k' * these conditions are

$$\sum\left[\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right]=0. \tag{14}$$

If, therefore, the lines $\rho_1, \rho_2, \ldots, \rho_{n-2}$ are coordinate lines of curvature on M_{n-2}, the functions α_i and β_i must satisfy the conditions (12) and (13′). These conditions being satisfied, we shall show that F is also a solution of (12). We have from (11),

$$\begin{aligned}\frac{\partial^2 F}{\partial\rho_{k'}\partial\rho_k} &= \sum\frac{\partial p_i}{\partial\rho_{k'}}\frac{\partial\alpha_i}{\partial\rho_k}+\sum\frac{\partial\bar{q}_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\sum\bar{p}_i\frac{\partial^2\alpha_i}{\partial\rho_k\partial\rho_{k'}}+\sum\bar{q}_i\frac{\partial^2\beta_i}{\partial\rho_k\partial\rho_{k'}}\\ &=\lambda_{k'}\sum\left(\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right)+\sum\bar{p}_i\frac{\partial^2\alpha_i}{\partial\rho_k\partial\rho_{k'}}+\sum\bar{q}_i\frac{\partial^2\beta_i}{\partial\rho_k\partial\rho_{k'}},\end{aligned}$$

hence

$$\begin{aligned}(\lambda_k-\lambda_{k'})\frac{\partial^2 F}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_k}\frac{\partial F}{\partial\rho_{k'}}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial F}{\partial\rho_{k'}} &=\lambda_{k'}(\lambda_k-\lambda_{k'})\sum\left(\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}+\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}\right)\\ &+\sum\bar{p}_i\left[(\lambda_k-\lambda_{k'})\frac{\partial^2\alpha_i}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\alpha_i}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\alpha_i}{\partial\rho_{k'}}\right]\\ &+\sum\bar{q}_i\left[(\lambda_k-\lambda_{k'})\frac{\partial^2\beta_i}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\alpha_i}{\partial\rho_{k'}}\right].\end{aligned}$$

Now, if the condition of integrability (12) and (14) are satisfied, the right side of this equation vanishes identically; hence F is a solution of (12) *q. e. d.* It may be worth while to note that the system (12) is also satisfied by the function $\Sigma\alpha_i\beta_i$, α_i and β_i being a set of particular solutions. We have then the following:

THEOREM I. *If on a surface* (6) *the lines of curvature are coordinate lines, the curvilinear coordinates* α_i, β_i *must satisfy the system of differential equations,*

$$(\lambda_k-\lambda_{k'})\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\theta}{\partial\rho_{k'}}=0, \tag{12}$$

and also the $\dfrac{n-2\cdot n-3}{2}$ *relations*

$$\sum\left(\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}+\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}\right)=0. \tag{14}$$

*This amounts to assuming that the surface (6) is an $n-2$-spread. If $\lambda_k=\lambda_{k'}$ for *all* values of k and k', the surface is one-dimensional or a curve.

We shall transform the coordinates α_i and β_i putting

$$\left.\begin{array}{l}\alpha_1+\beta_1=2y_1,\quad \alpha_2+\beta_2=2y_3,\ \ldots,\ \alpha_{\frac{n-2}{2}}+\beta_{\frac{n-2}{2}}=2y_{n-3},\\ i(\alpha_1-\beta_1)=2y_2,\quad i(\alpha_2-\beta_2)=2y_4,\ \ldots,\ i(\alpha_{\frac{n-2}{2}}-\beta_{\frac{n-2}{2}})=2y_{n-2}.\end{array}\right\}\quad (16)$$

The equations (10) now become

$$\frac{1}{2}\frac{\partial}{\partial\rho_k}(\bar{p}_i+\bar{q}_i)=\lambda_k\frac{\partial y_{2i-1}}{\partial\rho_k},\quad \frac{i}{2}\frac{\partial}{\partial\rho_k}(\bar{q}_i-\bar{p}_i)=\lambda_k\frac{\partial y_{2i}}{\partial\rho_k},\quad i=1,2,\ldots,\frac{n-2}{2},\quad (17)$$

and the new coordinates y_i are still solutions of (12). The conditions (14) are

$$\sum\frac{\partial y_i}{\partial\rho_k}\frac{\partial y_i}{\partial\rho_{k'}}=0,\qquad k,k'=1,2,\ldots,n-2,\ k\neq k'.\quad (18)$$

We write (12) in the form

$$\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}-\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}-\frac{1}{H_{k'}}\frac{\partial H_{k'}}{\partial\rho_k}\frac{\partial\theta_k}{\partial\rho_{k'}}=0,\quad (19)$$

where the H's are defined by the equations

$$\left.\begin{array}{l}\dfrac{\partial\lambda_k}{\partial\rho_{k'}}=-(\lambda_k-\lambda_{k'})\dfrac{1}{H_k}\dfrac{\partial H_k}{\partial\rho_{k'}},\\[2mm] \dfrac{\partial\lambda_{k'}}{\partial\rho_k}=\quad(\lambda_k-\lambda_{k'})\dfrac{1}{H_{k'}}\dfrac{\partial H_{k'}}{\partial\rho_k}.\end{array}\right\}\quad (20)$$

Observing that (18) expresses the $\frac{n-2\,.\,n-3}{2}$ conditions that the y's, considered as functions of the ρ's, shall form a completely orthogonal system in S_{n-2}, we put

$$dy_1^2+dy_2^2+\ldots+dy_{n-2}^2=H_1^2d\rho_1^2+H_2^2d\rho_2^2+\ldots+H_{n-2}^2d\rho_{n-2}^2\quad (21)$$

where $H_1,\ldots,H_{n-2}$ satisfy the $\frac{n-2\cdot n-3\cdot n-4}{2}$ conditions (Darboux, *loc. cit.*, p. 165).

$$f_{kk'}(H_{k''kk'})=\frac{\partial^2H_{k''}}{\partial\rho_k\partial\rho_{k'}}-\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}}\frac{\partial H_{k''}}{\partial\rho_k}-\frac{1}{H_{k'}}\frac{\partial H_{k'}}{\partial\rho_k}\frac{\partial H_{k''}}{\partial\rho_{k'}}=0.\quad (22)$$

$n-2$ such functions being found, the functions λ_k may be found by integrating the system (20); $\bar{p}_i$, $\bar{q}_i$ and F are then determined by quadratures from (17) and (11). The systems (20), which determine the λ's, are equivalent to a single system

$$\frac{\partial\lambda_k}{\partial\rho_{k'}}=-(\lambda_k-\lambda_{k'})\frac{1}{H_k}\frac{\partial H_k}{\partial\rho_{k'}},$$

which admits of one and only one set of solutions $\lambda_1\ldots\lambda_{n-2}$, such that

$\lambda_k = f_k(\rho_k)$ for $\rho_k = \rho_k^0$ where ρ_k^0 is an initial value of ρ_k (Darboux, *loc. cit.*, p. 350). We have then

THEOREM II. *The determination of a surface in S_{n-1} on which the lines of curvature are coordinate lines depends on the determination of a completely orthogonal system in a space S_{n-2} of one dimension less. Such a system being given the corresponding surface is found by integrating the system*

$$\frac{\partial \lambda_k}{\partial \rho_{k'}} = -(\lambda_k - \lambda_{k'}) \frac{1}{H_k} \frac{\partial H_k}{\partial \rho_{k'}}, \tag{20'}$$

and by quadratures.

The first part of this theorem has been proved by Darboux* by a different method, starting with the generalized Olinde Rodrigue's formulae for n-space. The method used here will be of advantage in the transformation to the space $\bar{\bar{S}}_{n-2}$.

Instead of determining a system of λ's from (20′), we may take any solution whatever of (19). Let Θ be such a solution; then putting $F = \Theta$ we have at once a surface on which the ρ's are coordinate lines of curvature. In case a general solution of (19) can be found, and such solution always exists since the conditions (22) are satisfied, we obtain all the surfaces having the same spherical representation of the lines of curvature.

APPLICATIONS.

§ 3. Let $H_i = 1$. Then $\dfrac{\partial \lambda_k}{\partial \rho_k} = 0$, or, $\lambda_k = \phi_k''(\rho_k)$. Equations (19) are now

$$\frac{\partial^2 \theta}{\partial \rho_k \partial \rho_{k'}} = 0, \tag{23}$$

the general solution of which is

$$\theta = \phi_1 + \phi_2 + \ldots + \phi_{n-2},$$

ϕ_i being a function of ρ_i alone. Taking as particular solutions of (23) the $n-2$ independent functions $y_i = \rho_i$ we have by integrating the system (17),

$$\bar{q}_i + \bar{p}_i = 2\phi_{2i-1}', \quad i(\bar{q}_i - \bar{p}_i) = 2\phi_{2i}', \qquad i = 1, 2, \ldots, \frac{n-2}{2},$$

and integrating (11) we find

$$F = 2\sum_1^{n-2} \phi_i + 2c,$$

* Darboux, *loc. cit.*, pp. 178–182.

which shows that F is a general solution of (23). The tangential equation of the surface is, therefore,

$$2\rho_1 X_1 + 2\rho_2 X_2 + \ldots + (1-\Sigma\rho_i^2) X_{n-1} + 2\Sigma\phi_i + 2c = 0. \tag{24}$$

Substituting the values of the α's in terms of the ρ's in (6) we have the surface

$$\left.\begin{aligned} &X_{2i-1} = -\phi'_{2i-1} + \rho_{2i-1} X_{n-1}, \quad X_{2i} = -\phi'_{2i} + \rho_{2i} X_{n-1}, \\ &X_{n-1} = \frac{2\Sigma\phi'_{2i-1}\rho_{2i-1} + 2\Sigma\phi'_{2i}\rho_{2i} - 2\Sigma\phi_i - 2c}{1+\Sigma\rho_i^2}, \\ &\qquad\qquad i = 1, 2, \ldots, \frac{n-2}{2}. \end{aligned}\right\} \tag{25}$$

These equations show that the surfaces have their lines of curvature plane in all the $n-2$ systems. The spherical representation of these lines are circles passing through the point $x_1 = x_2 = \ldots = x_{n-2} = 0$, $x_{n-1} = 1$, as readily appears from (24). The focal surface has $n-2$ sheets, viz.:

$$\overline{X}_i = \rho_i \phi''_k - \phi'_i, \quad \overline{X}_{n-1} = \frac{1-\Sigma\rho_i^2}{2}\phi''_k + \Sigma\rho_i\phi'_i - \Sigma\phi_i - c, \qquad i, k = 1, 2, \ldots, n-2,$$

where the subscript k indicates the individual sheet.*

As a particular case let us put $\phi_{2i-1} = \dfrac{k+2}{4}\rho_{2i-1}^2$, $\phi_{2i} = \dfrac{k-2}{4}\rho_{2i}^2$, $c = \dfrac{k}{4}$. Substituting in (24) and differentiating with respect to $\rho_1, \rho_2 \ldots \rho_{n-2}$ in succession, we have

$$2X_{2i-1} - 2\rho_{2i-1}X_{n-1} = -(k+2)\rho_{2i-1}, \quad 2X_{2i} - 2\rho_{2i}X_{n-1} = -(k-2)\rho_{2i}. \tag{27}$$

Eliminating the ρ's from (24) and (27) we have the cubic surface

$$2X_{n-1}\sum_1^{n-1} X_i^2 - (k-2)\sum_1^{\frac{n-2}{2}} X_{2i-1}^2 - (k+2)\sum_1^{\frac{n-2}{2}} X_{2i}^2 - kX_{n-1}^2 + \frac{k^2-4}{2}X_{n-1} + \frac{k(k^2-4)}{4} = 0, \tag{27'}$$

a special kind of cyclide which we shall study later on.

---

*The equation of the focal surface may be obtained from equations (19) A, p. 207, by introducing the variables y, instead of the α's and β's. We obtain the following:

$$\overline{X}_i = \frac{y_i\sigma_k}{2} - \frac{p_i}{2}, \quad \overline{X}_{n-1} = \frac{1-\Sigma y_i^2}{4}\sigma_k + \frac{1}{2}\Sigma p_i y_i - \frac{F}{2}, \qquad i, k = 1, 2, \ldots, n-2,$$

where σ_k is a root of the equation

$$\begin{vmatrix} p_{11}-\sigma & p_{12} \cdots\cdots & p_{1n-2} \\ p_{21} & p_{22}-\sigma \cdots & p_{2n-2} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ p_{n-21} & p_{n-22} \cdots & p_{n-2n-2}-\sigma \end{vmatrix} = 0. \tag{26}$$

Let H be of the form $\frac{1}{h}$, h being a function of the ρ's. Darboux has shown * that h must be of the form

$$a(\rho_1^2+\rho_2^2+\ldots+\rho_{n-2}^2)+2a_1\rho_1+2a_2\rho_2+\ldots+a_{n-2}\rho_{n-2}+b,$$

where the coefficients satisfy the relation $\Sigma a_i^2=ab$. The coordinates y_i expressed in terms of the ρ's are:

$$y_1=\frac{\rho_1+\frac{a_1}{a}}{h},\ \ldots,\ y_{n-2}=\frac{\rho_{n-2}+\frac{a_{n-2}}{a}}{h}. \tag{28}$$

The differential equations for determining the λ's are

$$\frac{\partial\lambda_k}{\partial\rho_{k'}}=\frac{2(\lambda_k-\lambda_{k'})(a\rho_{k'}+a_{k'})}{h},$$

the general integral of which is

$$\lambda_k=h\phi_k''-2\sum_1^{n-2}\phi_i'(a\rho_i+a_i)+2a\sum_1^{n-2}\phi_i;$$

integrating the systems (17) and (19) we find that F is of the form $F=\frac{2\Sigma\phi_i}{h}$, a result which might also have been obtained by integrating the system of differential equations

$$\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}+\frac{2(a\rho_{k'}+a_{k'})}{h}\frac{\partial\theta}{\partial\rho_k}+\frac{2(a\rho_k+a_k)}{h}\frac{\partial\theta}{\partial\rho_{k'}}=0,$$

of which the y's are particular solutions. The tangential equation of the surface is

$$2\Big(\rho_1+\frac{a_1}{a}\Big)X_1+2\Big(\rho_2+\frac{a_2}{a}\Big)X_2+\ldots+\Big(h-\frac{1}{a}\Big)X_{n-1}+2\Sigma\phi_i=0, \tag{24'}$$

from which it appears that no new surfaces are obtained, since (24) may be transformed into (24′) by taking as new ρ's linear functions of the old such that $\rho_i=-(a\rho_i'+a_i)$. It may also be observed that the equations (28) define an inversion in the space S_{n-2}.

§ 4. *Surfaces whose Lines of Curvature are Plane in all Systems.*

The surfaces thus far obtained are not the only ones having their lines of curvature plane in all the $n-2$ systems. The spherical representation of the lines of curvature of such surfaces must consist of an $n-2$-fold orthogonal system of circles on the sphere. The orthogonal system corresponding to the lines of curvature on the surface (23) is a special one, the circles lying in $n-2$

* Darboux, *loc. cit.*, pp. 166–168.

systems of planes which meet at $x_1=x_2=\ldots\ldots=x_{n-2}=0$, $x_{n-1}=1$ and all the circles of any one system being tangent to each other at this point. The $n-2$ lines of intersection of the planes of the $n-2$ systems are parallel to the coordinate axes $x_1, x_2, \ldots\ldots, x_{n-2}$ and perpendicular to the x_{n-1} axis. We may obtain a general system in the following manner: We pass an $n-r$-fold system of $(r-1)$-flats through a point on the x_{n-1}-axis at a distance from the origin $=a$, all the flats of the system having as common axis the flat $x_1=x_2=\ldots\ldots=x_{n-r}=0$, $x_{n-1}=a$. We also pass an $r-2$-fold system of $(n-r+1)$-flats through a point on the x_{n-1}-axis at a distance $\frac{1}{a}$ from the origin, the common axis being the flat $x_{n-r+1}=x_{n-r+2}=\ldots\ldots=x_{n-2}=0$, $x_{n-1}=\frac{1}{a}$. *We shall prove that these two systems of flats will determine on the sphere an $n-2$-fold orthogonal system of circles.* This proposition is an extension to n-space of an analogous one for 3-space. If $a=1$ we get the spherical representation of lines of curvature of the surfaces (24), and for $a=0$ the system is analogous to that of meridians and circles of latitude on a sphere in 3-space.

We express the coordinates x_i of the sphere in terms of the curvilinear coordinates θ_i as follows:

$$\left.\begin{aligned}
x_1&=\frac{\sqrt{1-a^2}\sin\theta_{n-2}\sin\theta_{n-r}\ldots\ldots\sin\theta_2\sin\theta_1}{1+a\sin\theta_{n-2}\cos_{n-r}},\\
x_2&=\frac{\sqrt{1-a^2}\sin\theta_{n-2}\sin\theta_{n-r}\ldots\ldots\sin\theta_2\cos\theta_1}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-r}&=\frac{\sqrt{1-a^2}\sin\theta_{n-2}\sin\theta_{n-r}\cos\theta_{n-r-1}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
x_{n-r+1}&=\frac{\sqrt{1-a^2}\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\ldots\cos\theta_{n-3}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-2}&=\frac{\sqrt{1-a^2}\cos\theta_{n-2}\sin\theta_{n-r+1}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\quad x_{n-1}=\frac{a+\cos\theta_{n-r}\sin\theta_{n-2}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},
\end{aligned}\right\}\quad(29)$$

from which it is easily verified that $\Sigma x_i^2=1$. In order to prove that the system is orthogonal we need only calculate the linear element $d\sigma$. A rather long, but not inherently difficult calculation will show that this element is

$$\begin{aligned}d\sigma^2=E_1d\theta_1^2+E_2d\theta_2^2+\ldots\ldots+E_{n-r}d\theta_{n-r}^2+E_{n-r+1}d\theta_{n-r+1}^2\\
+\ldots\ldots+E_{n-3}d\theta_{n-3}^2+E_{n-2}d\theta_{n-2}^2,\quad(30)\end{aligned}$$

where the E's have the following values:

$$\left.\begin{aligned}
E_1 &= \frac{(1-a^2)\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_2}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2},\\
E_2 &= \frac{(1-a^2)\sin^2\theta_{n-2}\sin^2\theta_{n-r}\ldots\sin^2\theta_3}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
E_{n-r} &= \frac{(1-a^2)\sin^2\theta_{n-2}}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2}, \quad E_{n-r+1} = \frac{(1-a^2)\cos^2\theta_{n-2}}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2},\\
E_{n-r+2} &= \frac{(1-a^2)\cos^2\theta_{n-2}\cos^2\theta_{n-r+1}}{(1+a\sin\theta_{n-2}\cos_{n-r})^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
E_{n-3} &= \frac{(1-a^2)\cos^2\theta_{n-2}\cos^2\theta_{n-r+1}\ldots\cos^2\theta_{n-4}}{(1+a\sin\theta_{n-2}\cos_{n-r})^2},\\
E_{n-2} &= \frac{(1-a^2)}{(1+a\sin\theta_{n-2}\cos\theta_{n-r})^2}.
\end{aligned}\right\} \quad (31)$$

That the curves θ_i are circles appears from the following systems of equations deduced from (29):

$$\left.\begin{aligned}
x_1 &= \frac{1}{\sqrt{1-a^2}}\,\frac{\sin\theta_{n-r}\ldots\sin\theta_2\sin\theta_1}{\cos\theta_{n-r}}(x_{n-1}-a),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-r} &= \frac{1}{\sqrt{1-a^2}}\,\frac{\sin\theta_{n-r}\cos\theta_{n-r-1}}{\cos\theta_{n-r}}(x_{n-1}-a),\\
x_{n-r+1} &= -\frac{a\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}}{\sqrt{1-a^2}}\left(x_{n-1}-\frac{1}{a}\right),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
x_{n-2} &= -\frac{a\cos\theta_{n-2}\sin\theta_{n-r+1}}{\sqrt{1-a^2}}\left(x_{n-1}-\frac{1}{a}\right),
\end{aligned}\right\} \quad (32)$$

and that the geometrical construction given at the beginning of this section is correct appears at once from these equations.

Since the coordinates y_i in terms of the x's are given by the formulae:

$$y_1 = \frac{x_1}{1+x_{n-1}}, \quad y_{n-2} = \frac{x_{n-2}}{1+x_{n-1}}, \quad x_1 = \frac{2y_1}{1+\Sigma y_i^2}, \ldots, x_{n-1} = \frac{1-\Sigma y_i^2}{1+\Sigma y_i^2}, \quad (33)$$

we have

$$\left.\begin{aligned}
y_1 &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\sin\theta_{n-2}\sin\theta_{n-r}\ldots\sin\theta_2\sin\theta_1}{1+\cos_{n-r}\sin\theta_{n-2}},\\
y_2 &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\sin\theta_{n-2}\sin\theta_{n-r}\ldots\cos\theta_1}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
y_{n-r} &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\sin\theta_{n-2}\sin\theta_{n-r}\cos\theta_{n-r-1}}{1+\cos_{n-r}\sin\theta_{n-2}},\\
y_{n-r+1} &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
y_{n-2} &= \frac{\sqrt{1-a^2}}{1+a}\,\frac{\cos\theta_{n-2}\sin\theta_{n-r+1}}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\\
1-\Sigma y^2 &= \frac{2(a+\cos\theta_{n-r}\sin\theta_{n-2})}{(1+a)(1+\cos\theta_{n-r}\sin\theta_{n-2})},\\
\frac{2}{1+\Sigma y^2} &= \frac{(a+1)(1+\cos\theta_{n-r}\sin\theta_{n-2})}{1+a\cos\theta_{n-r}\sin\theta_{n-2}}.
\end{aligned}\right\}\quad(34)$$

The coordinates y_i satisfy the following system of $\dfrac{n-2\cdot n-3}{2}$ differential equations:

$$\left.\begin{aligned}
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{k'}}-\cot\theta_{k'}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k,k'=1,2,\ldots,n-r-1),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{k'}}-\tan\theta_{k'}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k,k'=n-r+1,\ldots,n-r),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-r}}-\frac{\cot\theta_{n-r}+\csc\theta_{n-r}\sin\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k=1,2,\ldots,n-r-1),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{k'}}=0,\quad\begin{Bmatrix}k=1,2,\ldots,n-r-1\\ k'=n-r+1,\ldots,n-3\end{Bmatrix},\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-r}}+\frac{\cot\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k=n-r+1,\ldots,n-3),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-2}}-\frac{\cot\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_k}=0,\quad(k=1,2,\ldots,n-r+1),\\
&\frac{\partial^2\phi}{\partial\theta_k\partial\theta_{n-2}}+\frac{\tan\theta_{n-2}+\cos\theta_{n-r}\sec\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-3}}\frac{\partial\phi}{\partial\theta}=0,\quad(k=n-r+1,\ldots,n-3),\\
&\frac{\partial^2\phi}{\partial\theta_{n-r}\partial\theta_{n-2}}-\frac{\sin\theta_{n-2}\sin\theta_{n-r}}{1+\cos_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_{n-2}}-\frac{\cot\theta_{n-2}}{1+\cos\theta_{n-r}\sin\theta_{n-2}}\frac{\partial\phi}{\partial\theta_{n-r}}=0.\quad k\neq k'.
\end{aligned}\right\}\quad(35)$$

The general solution of this system is

$$F=\frac{\Sigma\Phi_i\sqrt{E_i}(1+\Sigma y_i^2)}{1+\cos\theta_{n-r}\sin\theta_{n-2}},\tag{36}$$

where Φ_i is a function of θ_i alone and the E's have the values given in (31). The surface is therefore the envelope of the flats

$$2y_1X_1+2y_2X_2+\dots+2y_{n-r}X_{n-r}+2y_{n-r+1}X_{n-r+1}+\dots \\ +(1-\Sigma y_i^2)X_{n-1}=(1+\Sigma y_i^2)\Sigma\Phi_i\sqrt{E_i}. \quad (37)$$

For any constant value of r all the surfaces obtained by putting for Φ_i arbitrary functions of θ_i have the same spherical representation of their lines of curvature. Since r can have any value from 3 to $n-1$ *there will be* $n-3$ *different types.* To these types must also be added the type (24) corresponding to $a=1$ obtained in § 3, viz.:

$$2\rho_1X_1+2\rho_2X_2+\dots+2\rho_{n-r}X_{n-r}+\dots+(1-\Sigma\rho_i^2)X_{n-1}+2\Sigma\phi_i+2c=0. \quad (24)$$

The surfaces (37) may be considered as the envelope of the radical flats of the two spheres:

$$\left.\begin{aligned}
&\sum_1^{n-1}X_i^2+2\cot\theta_{n-2}\cos\theta_{n-r+1}\dots\cos\theta_{n-3}X_{n-r+1}+\dots\\
&\quad+2\cot\theta_{n-2}\sin\theta_{n-r+1}X_{n-2}+\frac{2a\csc\theta_{n-2}}{\sqrt{1-a^2}}X_{n-1}=2\Phi_{n-2}\csc\theta_{n-2}\\
&\quad+2\Phi_{n-3}\cot\theta_{n-2}\dots\cos\theta_{n-4}+\dots+2\Phi_{n-r+2}\cot\theta_{n-2}\cos\theta_{n-r+1}\\
&\quad+2\Phi_{n-r+1}\cot\theta_{n-2}+C,\\
&\sum_1^{n-1}X_i^2-2\sin\theta_{n-r}\dots\sin\theta_1X_1-\dots-2\sin\theta_{n-r}\cos\theta_{n-r+1}X_{n-r}\\
&\quad-\frac{2\cos\theta_{n-r}}{\sqrt{1-a^2}}X_{n-1}=-2\Phi_{n-r}-2\Phi_{n-r-1}\sin\theta_{n-r}-\dots\\
&\quad-2\Phi_1\sin\theta_{n-r}\dots\sin\theta_2+C,
\end{aligned}\right\} \quad (38)$$

the centres of which lie respectively on the hyperboloid of revolution

$$X_{n-r+1}^2+X_{n-r+2}^2+\dots+X_{n-2}^2-\frac{X_{n-1}^2}{\dfrac{a^2}{1-a^2}}=-1,\ X_i=0,\ \ i=1,2,\dots,n-r, \quad (39)$$

and on the ellipsoid

$$\sum_1^{n-r}X_i^2+\frac{X_{n-1}^2}{\dfrac{1}{1-a^2}}=1, \qquad X_{n-r+1}=0,\ \dots,\ X_{n-2}=0. \quad (40)$$

These quadrics are focal, the vertex of one passing through the focus of the other. They are, moreover, perpendicular to each other, having the X_{n-1}-axis in common.*

---

*The two spaces in which the quadrics are immersed exhibit the maximum of perpendicularity expressed by the fraction $\frac{r-2}{r-1}$; See Schoute, *Mehrdimensionale Geometrie*, 1. Theil, p. 49.

The surfaces (24) may also be considered as the envelope of the two spheres

$$\left.\begin{aligned} &\text{a)}\ \Sigma X_i^2+4\rho_1X_1+\ldots\ldots+4\rho_{n-r}X_{n-r}+(1-2\sum_1^{n-r}\rho_i^2)X_{n-1}\\ &\qquad\qquad\qquad\qquad=-4\phi_1-\ldots\ldots-4\phi_{n-r}+C-2c,\\ &\text{b)}\ \Sigma X_i^2-4\rho_{n-r+1}X_{n-r+1}-\ldots\ldots-4\rho_{n-2}X_{n-2}-(1-2\sum_{n-r+1}^{n-2}\rho_i^2)X_{n-1}\\ &\qquad\qquad\qquad\qquad=4\phi_{n-r+1}+\ldots\ldots+4\phi_{n-2}+C+2c,\end{aligned}\right\}\quad(41)$$

the centres of which lie on the paraboloids

$$\left.\begin{aligned} &2X_{n-1}=2-\frac{\sum\limits_1^{n-r}X_i^2}{4},\quad X_i=0;\qquad(i=n-r+1,\ldots\ldots,n-2),\\ &2X_{n-1}=2-\frac{\sum\limits_{n-r+1}^{n-2}X_i^2}{4},\quad X_i=0;\qquad(i=1,2,\ldots\ldots,n-r).\end{aligned}\right\}\quad(42)$$

§ 5. *The Generalized Dupin Cyclides in* S_{n-1}.

If in (37) we put $\Phi_1=\Phi_2=\Phi_3=\ldots\ldots=\Phi_{n-r-1}=0$, $\Phi_{n-r}=1-\frac{ka}{\sqrt{1-a^2}}\cos\theta_{n-r}$, $\Phi_{n-r+1}=\Phi_{n-r+2}=\ldots\ldots=\Phi_{n-3}=0$, $\Phi_{n-2}=-\frac{k}{\sqrt{1-a^2}}$, and then derive the envelope of the resulting flat in the usual way we get the surface in the following parametric form:

$$\left.\begin{aligned} X_1&=\frac{\sin\theta_{n-r}\ldots\ldots\sin\theta_2\sin\theta_1(1-k\sqrt{1-a^2}\sin\theta_{n-2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\ X_2&=\frac{\sin\theta_{n-r}\ldots\ldots\sin\theta_2\cos\theta_1(1-k\sqrt{1-a^2}\sin\theta_{n-2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ X_{n-r}&=\frac{\sin\theta_{n-r}\cos\theta_{n-r-1}(1-k\sqrt{1-a^2}\sin\theta_{n-2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\ X_{n-r+1}&=\frac{-\cos\theta_{n-2}\cos\theta_{n-r+1}\ldots\ldots\cos\theta_{n-4}(a\cos\theta_{n-r}+k\sqrt{1-a^2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ X_{n-2}&=-\frac{\cos\theta_{n-2}\sin\theta_{n-r+1}(a\cos\theta_{n-r}+k\sqrt{1-a^2})}{1+a\sin\theta_{n-2}\cos\theta_{n-r}},\\ X_{n-1}&=\frac{\sqrt{1-a^2}}{a}\cdot\frac{a\cos\theta_{n-r}+k\sqrt{1-a^2}}{1+a\sin\theta_{n-2}\cos\theta_{n-r}}-\frac{k}{a}.\end{aligned}\right\}\quad(43)$$

Eliminating the parameters θ_i we obtain the quartic surface

$$(1-a^2)[\sum_1^{n-1}X_i^2-k^2-1]^2=4\left[a^2\left(X_{n-1}+\frac{k}{a}\right)^2-(1-a^2)(\sum_{n-r+1}^{n-2}X_i^2)\right],\quad(44)$$

which, as we shall prove, is a generalization of the Dupin cyclide of ordinary space. The focal sheets of the surface are given by the formulae

$$\overline{X}_i = X_i + R_k x_i,$$

where R_k are the $n-2$ principal radii of curvature calculated from the equations of Olinde Rodrigue:

$$\frac{\partial X_i}{\partial \theta_k} + R_k \frac{\partial x}{\partial \theta_k}, \quad \begin{Bmatrix} i=1, 2, \ldots, n-1 \\ k=1, 2, \ldots, n-2 \end{Bmatrix},$$

where x_i have the values given in (29). Although a general surface (37) will have $n-2$ different focal sheets, in the case of the surface (43) it is found that *only two focal sheets exist*, viz.,

$$\left.\begin{aligned} &\text{a)}\ X_1^2+X_2^2+\ldots+X_{n-r}^2+\frac{X_{n-1}^2}{\dfrac{1}{1-a^2}}=1,\ X_i=0, \quad i=n-r+1, \ldots, n-2, \\ &\text{b)}\ X_{n-r+1}^2+X_{n-r+2}^2+\ldots+X_{n-2}^2-\frac{X_{n-1}^2}{\dfrac{a}{1-a^2}}=-1,\ X_i=0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1, 2, \ldots, n-r, \end{aligned}\right\} \quad (45)$$

a pair of focal quadrics, of which one is an ellipsoid immersed in the space $X_{n-r+1}=X_{n-r+2}=\ldots=X_{n-2}=0$, and the other a hyperboloid immersed in the space $X_1=X_2=\ldots=X_{n-r}=0$. If in (38) we put $\Phi_1=\Phi_2=\ldots=\Phi_{n-r-1}=0$, $\Phi_{n-r}=1-\dfrac{ak}{\sqrt{1-a^2}}\cos\theta_{n-r}$, $\Phi_{n-r+1}=\ldots=\Phi_{n-3}=0$, $\Phi_{n-2}=\dfrac{-k}{\sqrt{1-a^2}}$, and $C=1+k^2$, we have the two spheres,

$$\left.\begin{aligned} &\text{a)}\ \sum_1 X_i^2+2\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}X_{n-r+1}+\ldots \\ &\qquad +2\cot\theta_{n-2}\sin\theta_{n-r+1}X_{n-2}+\frac{2a\csc\theta_{n-2}}{\sqrt{1-a^2}}X_{n-1}=1+k^2-\frac{2k}{\sqrt{1-a^2}}\csc\theta_{n-2}, \\ &\text{b)}\ \Sigma X_i^2-2\sin\theta_{n-r}\ldots\sin\theta_1 X_1-\ldots-2\sin\theta_{n-r}\cos\theta_{n-r+1}X_{n-r} \\ &\qquad -\frac{2\cos\theta_{n-r}}{\sqrt{1-a^2}}X_{n-1}=1+k^2-2\left[1-\frac{ak}{\sqrt{1-a^2}}\cos\theta_{n-r}\right], \end{aligned}\right\} \quad (46)$$

the radii R_1 and R_2 of which are

$$R_1=\pm\left[\frac{\csc\theta_{n-2}}{\sqrt{1-a^2}}-k\right], \quad R_2=\pm\left[\frac{a}{\sqrt{1-a^2}}\cos\theta_{n-r}+k\right],$$

from which we derive

$$R_1+R_2=\pm\left[\frac{\cos\theta_{n-2}+a\cos\theta_{n-r}}{\sqrt{1-a^2}}\right]=D. \quad (47)$$

Hence, *the surface* (43) *is the envelope of two sets of spheres that touch, and whose centres lie respectively on the ellipsoid* (45a) *and hyperboloid* (45b). *The surface is thus a generalization of the Dupin cyclide in ordinary space.* If k varies we get parallel cyclides as appears from (47).

If we give to r in (43) the successive values 3, 4, ..., $n-1$ we obtain $n-3$ different types of cyclides.* Thus, if $n=3$ the focal hyperboloid becomes a hyperbola lying in the plane $X_1=X_2=\ldots=X_{n-3}=0$, while the ellipsoid is $n-3$ dimensional and is immersed in the space $X_{n-2}=0$. If S_{n-1} is an odd space, and $r=\frac{n+2}{2}$, the focal quadrics are of the same dimension $=\frac{n-2}{2}$, and the equation of the corresponding cyclide is

$$(1-a^2)\,[\Sigma X_i^2-k^2-1]^2 \\ =4\left[a^2\left(X_{n-1}+\frac{k}{a}\right)^2-(1-a^2)\,(X^2_{\frac{n}{2}}+X^2_{\frac{n+2}{2}}+\ldots+X^2_{n-2})\right]. \quad (48)$$

§ 6. We shall next discuss the type of cyclides for which $a=1$. In (24) we put $\phi_i=\frac{k+2}{4}\rho_i^2$, $i=1, 2, \ldots, n-r$, $\phi_{n-r+k}=\frac{k-2}{4}\rho^2_{n-r+k}$, $k=1, 2, \ldots, r-2$, $c=\frac{k}{4}$, and obtain the surface in the following parametric form:

$$\left.\begin{aligned} &2X_i-2\rho_iX_{n-1}+(k+2)\rho_i=0, \quad i=1, 2, \ldots, n-r, \\ &2X_{n-r+k}-2\rho_{n-r+k}X_{n-1}+(k-2)\rho_{n-r+k}=0, \quad k=1, 2, \ldots, r-2, \\ &X_{n-1}=\frac{\frac{k+2}{2}\sum\limits_1^{n-r}\rho_i^2+\frac{k-2}{2}\sum\limits_{n-r+1}^{n-2}\rho_i^2-\frac{k}{2}}{1+\Sigma\rho_i^2}, \end{aligned}\right\} \quad (49)$$

or, in Cartesian form,

$$2X_{n-1}\left(\sum_1^{n-1}X_i^2+\frac{k^2-4}{4}\right)-(k-2)\sum_1^{n-r}X_i^2 \\ -(k+2)\sum_{n-r+1}^{n-2}X_i^2-kX^2_{n-1}+\frac{k(k^2-4)}{4}=0. \quad (49')$$

---

*The types for which $r>\frac{n+2}{2}$ do not differ essentially from those for which $r<\frac{n+2}{2}$; the focal ellipsoid and hyperboloid merely interchange. We need therefore consider only the surfaces of type $r \overline{<} \frac{n+2}{2}$.

The two sets of spheres (41) are, putting $C=\frac{k^2}{4}$,

$$\left.\begin{aligned}
&\sum_1^{n-1} X_i^2+4\rho_1X_1+\dots+4\rho_{n-r}X_{n-r}+\Big(1-2\sum_1^{n-r}\rho_i^2\Big)X_{n-1}\\
&\qquad\qquad =-(k+2)\sum_1^{n-r}\rho_i^2+\frac{k^2}{4}-\frac{k}{2},\\
&\sum_1^{n-1} X_i^2-4\rho_{n-r+1}X_{n-r+1}-\dots-4\rho_{n-3}X_{n-3}-\Big(1-2\sum_{n-r+1}^{n-2}\rho_i^2\Big)X_{n-1}\\
&\qquad\qquad =(k-2)\sum_{n-r+1}^{n-2}\rho_i^2+\frac{k^2}{4}+\frac{k}{2},
\end{aligned}\right\} \tag{50}$$

and the radii are, respectively:

$$R_1=\pm\tfrac{1}{2}\Big(1+2\sum_1^{n-r}\rho_i^2-k\Big),\quad R_2=\pm\tfrac{1}{2}\Big(1+2\sum_{n-r+1}^{n-2}\rho_i^2+k\Big). \tag{51}$$

$\therefore R_1+R_2=\pm\Big(1+\sum_1^{n-2}\rho_i^2\Big)=$ Distance between the centres.

Hence, *the cyclide of the third order is the envelope of two sets of spheres that touch, and whose centres lie respectively on two focal paraboloids of revolution whose equations are*

$$\left.\begin{aligned}
&4X_{n-1}+2=\sum_{n-r+1}^{n-2}X_i^2,\quad X_1=X_2=X_{n-r}=0,\\
&4X_{n-1}-2=-\sum_1^{n-r}X_i^2,\quad X_{n-r+1}=X_{n-r+2}=\dots=X_{n-2}=0.
\end{aligned}\right\} \tag{52}$$

If $r=3$, one paraboloid is one-dimensional, *i. e.*, a parabola, while the other is of dimension $n-3$. If S_{n-1} is an odd space and $r=\frac{n+2}{2}$, both paraboloids are of the same dimension, viz.: $\frac{n}{2}$. All the cyclides obtained by putting $r=3,\dots,n=1$ in succession in (49) have the same spherical representation, while in the case of the hyperbolic-elliptic types we get a different orthogonal system of circles on the Gaussian sphere for each value of r.* If k varies we have a system of parallel cyclides as before.

---

*Schoute in his *Mehrdimensionale Geometrie*, II. Theil, pp. 316–320, has by a synthetic method borrowed from 3-space, derived the type for which $r=n-1$, *i. e.*, where the focal quadrics are an ellipse and a hyperboloid of revolution of $n-3$ dimensions. That this method leads to the n-dimensional generalization of Dupin's cyclide in the sense that we get *all* such cyclides is, however, not true, although the opposite might be inferred from the author's statement. On p. 320 the author mentions the existence of two families of spheres of the general nature considered above, but does not consider their envelope. The author's reference to a short article in *Verslagen der Akademie von Amsterdam*, vom Februar 1905, I have not been able to look up.

§ 7. *Some Geometrical Properties of the Cyclides in General Space.*

The loci on the cyclide where the radii of the two sets of spheres equal zero will be spreads of double points on the surface. We obtain these point-spheres by equating R_1 and R_2 to zero giving

$$\csc\theta_{n-2}=k\sqrt{1-a^2},\quad \cos\theta_{n-r}=-\frac{k}{a}\sqrt{1-a^2}.$$

Substituting the values of $\cot\theta_{n-2}$ and $\csc\theta_{n-2}$ in (43) and eliminating the remaining parameters θ_i, we get an $r-3$-dimensional sphere whose equations are

$$X_1=X_2=\ldots.=X_{n-r}=0,\quad \sum_{n-r+1}^{n-2} X_i^2=k^2(1-a^2)-1,\quad X_{n-1}=ka, \tag{53}$$

all the points of which must be considered as centres of null-spheres. Again, substituting the values of $\cos\theta_{n-r}$ and $\sin\theta_{n-r}$ in (43) we get a second locus of double points, viz.:

$$\sum_1^{n-r} X_i^2=\frac{a^2-k^2(1-a^2)}{a^2},\quad X_{n-r+1}=X_{n-r+2}=\ldots.=X_{n-2}=0,\quad X_{n-1}=\frac{k}{a}, \tag{54}$$

(53) lies on the focal hyperboloid H_{r-2} and (54) on the focal ellipsoid E_{n-r}. We shall denote these spheres by Σ_{r-3} and Σ_{n-r-1}, and the two systems of spheres by S_h and S_e, the spheres S_h having their centres on H_{r-2}, and the spheres S_e on E_{n-r}. Since all the point-spheres having their centres on Σ_{r-3} are tangents to all the spheres S_h, their centres must belong to these spheres, hence all the spheres S_h pass through the locus Σ_{r-3}; similarly all spheres S_e pass through the locus Σ_{n-r+1}. We may therefore say that *the quartic cyclides in S_{n-1} is the envelope of spheres having their centres on the ellipsoid E_{n-r} and passing through the sphere-locus Σ_{r-3} on H_{r-2}*; or, *it is the envelope of spheres having their centres on the hyperboloid H_{r-2} and passing through the sphere-locus Σ_{n-r-1} on the ellipsoid E_{n-r}*.

Consider the flat

$$u_1X_1+u_2X_2+\ldots.+u_{n-1}X_{n-1}+p=0; \tag{55}$$

the condition that it shall be tangent to H_{r-2} and E_{n-r} are

$$\sum_1^{n-r} u_i^2+\frac{1}{1-a^2}u_{n-1}^2=p^2,\quad -\sum_{n-r+1}^{n-2} u_i^2+\frac{a^2}{1-a^2}u_{n-1}^2=p^2, \tag{56}$$

from which it follows that we must have $\sum_1^{n-1} u_i^2=0$, which means that *the flat is isotropic.* Introducing in (55) the parameters

$$u_1=2\rho_1,\ \ldots.,\ u_{n-3}=2\rho_{n-3},\quad u_{n-2}=1-\Sigma\rho_i^2,\quad u_{n-1}=i(1+\Sigma\rho_i^2),$$

we obtain the ∞^{n-3} flats

$$2\rho_1 X_1 + 2\rho_2 X_2 + \dots + (1-\Sigma\rho_i^2) X_{n-2} + i(1+\Sigma\rho_i^2) X_{n-1} + p = 0. \tag{57}$$

where p has the value

$$p = \pm\sqrt{4\sum_1^{n-r}\rho_i^2 - \frac{(1+\Sigma\rho^2)^2}{1-a^2}},$$

The equations of the "developable" surface obtained by letting the ρ's vary are as follows:

$$\left.\begin{aligned} X_i &= \frac{-2i\rho_i}{1+\Sigma\rho^2} X_{n-1} + \frac{2a^2\rho_i}{p(1-a^2)}, \quad (i=1, 2, \dots, n-r), \\ X_i &= \frac{-2i\rho_i}{1+\Sigma\rho_i^2} X_{n-1} + \frac{2\rho_i}{p(1-a^2)}, \quad (i=n-r+1, \dots, n-3), \\ X_{n-2} &= -\frac{i(1-\Sigma\rho_i^2)}{1+\Sigma\rho_i^2} X_{n-1} + \frac{1-\Sigma\rho_i^2}{p(1-a^2)}. \end{aligned}\right\} \tag{58}$$

In a former paper* we have studied isotropic complexes of which (58) is a special case. Since the complex contains ∞^{n-3} isotropic lines we shall denote it by the symbol Δ_{n-3}. By a theorem proved in A. I., p. 207, we know that it has a focal surface which is the envelope of the ∞^{n-4} isotropic 2-spreads contained in it, and that this focal surface has in general $n-3$ sheets. The edges of regression on the surface are minimal curves satisfying a certain system of differential equations. In the special case of (58) it may be proved that *there are only three sheets;* in fact, the determinant equation (13), p. 205, l. c., which determines the number of sheets, reduces to the simple form

$$(\sigma-A)^{n-r-1}(\sigma-B)^{r-3}\left[\sigma - A - \frac{16a^2\sum_1^{n-r}\rho_i^2}{p^3(1-a^2)}\right] = 0, \tag{59}$$

where

$$A = \frac{4}{p} - \frac{2(1+\Sigma\rho_i^2)}{p(1-a^2)}, \quad B = \frac{-2(1+\Sigma\rho_i^2)}{p(1-a^2)}.$$

Substituting the values of σ in equations (60), footnote, we obtain the three focal sheets, *two of which are seen to be precisely the focal quadrics H_{r-3}* and

*A, §§ 1–5 where the formulae developed are true for any space, even or odd, if instead of the parameters α_i and β_i we use ρ_i (see II, § 1, of this paper). The equation determining the focal sheets becomes the equation (26), note, when written of order $n-3$. The equations of the focal sheets take the form:

$$\left.\begin{aligned} &X_i = \frac{\rho_i\sigma_k}{2} - \frac{p}{2}, \quad X_{n-1} = \frac{1-\Sigma\rho^2_i}{4}\sigma_k + \frac{1}{2}\Sigma p_i\rho_i - \frac{p}{2}, \\ &X_{n-1} = i\left[\frac{1+\Sigma\rho_i^2}{4}\sigma_k - \frac{1}{2}\Sigma p_i\rho_i + \frac{p}{2}\right], \quad \begin{Bmatrix} i=1, 2, \dots, n-3 \\ k=1, 2, 3 \end{Bmatrix}. \end{aligned}\right\} \tag{60}$$

E_{n-r}. The surface of reference of Δ_{n-3} is the imaginary ellipsoid in the space $X_{n-1}=0$:

$$\frac{(1-a^2)}{a^2}\sum_{1}^{n-r} X_i^2+(1-a^2)\sum_{n-r-1}^{n-2} X_i^2+1=0, \quad X_{n-1}=0 \tag{61}$$

obtained by putting $X_{n-1}=0$ in (58). We have thus proved the

THEOREM. *The tangent n—2-flats common to the two focal quadrics* H_{r-3} *and* E_{n-r} *generate an isotropic complex* Δ_{n-3} *which has* H_{n-3} *and* E_{n-r} *for two of its three focal sheets and the imaginary quadric* (61) *for surface of reference.*

We shall now return to the study of the quartic cyclide (44). Any straight line $\overline{BA}$ joining a point A on Σ_{r-4} to a point B on Σ_{n-r-1} is an isotropic line. In fact, from (53) and (54) we find that the distance is:

$$D=\sqrt{k^2(1-a^2)-1+\frac{a^2-k^2(1-a^2)}{a^2}+k^2\left(a-\frac{1}{a}\right)^2}=0.$$

The ∞^{n-4} isotropic lines generate a locus D_{n-4} which is the intersection of the quadric cylinders

$$\sum_{1}^{n-r} X_i^2=\frac{a^2r_2^2}{k^2(1-a^2)^2}[X_{n-1}+ak]^2, \quad \sum_{n-r+1}^{n-2} X_i^2=\frac{a^2r_1^2}{k^2(1-a^2)}\left[X_{n-1}+\frac{k}{a}\right]=0.\text{*} \tag{62}$$

If we substitute the values of $\sum\limits_{1}^{n-r} X_i^2$ and $\sum\limits_{n-r+1}^{n-2} X_i^2$ from these equations in (44) we find that it is identically satisfied, hence *the* ∞^{n-4} *isotropic lines* $\overline{AB}$ *lie on the cyclide.* But they also belong to the isotropic complex Δ_{n-3}, hence

THEOREM. *The quartic cyclides in* S_{n-1} *is inscribed in the isotropic complex* Δ_{n-3}, *the locus of contact being the intersection of two quadric cylinders* (62).

Since the lines $\overline{AB}$ are lines of contact between the point-spheres on Σ_{r-4} and Σ_{n-r-1} and the cyclides, they are lines of curvature.

Consider any two points A and B on Σ_{r-3} and Σ_{n-r-1}. The two tangent flats to H_{r-2} and E_{n-r} at A and B respectively intersect the X_{n-1}-axis in a point C whose coordinates are:

$$X_1=X_2=\ldots=X_{n-2}=0, \quad X_{n-1}=\frac{-a}{k(1-a^2)}.$$

The sphere having its centre at this point and radius equal to $\overline{CB}$ (or $\overline{CA}$) is

$$\Sigma X_i^2+\left[X_{n-1}+\frac{a}{k(1-a^2)}\right]^2=\frac{[1-k^2(1-a^2)][a^2-k^2(1-a^2)]}{k^2(1-a^2)^2}, \tag{63}$$

* r_1 and r_2 in these equations are the radii of the spheres (53) and (54).

which, as is easily seen from equations (53) and (54), contains the loci of double points Σ_{r-3} and Σ_{n-r-1}. But the isotropic line $\overline{BA}$ is normal to the radius $\overline{CB}$ (or $\overline{CA}$), and, since it is isotropic, it must lie on the sphere. Hence, the sphere (64) is tangent to the cyclide along the ∞^{n-4} isotropic lines $\overline{AB}$. We shall call this sphere *the principal sphere.**

The point-spheres on Σ_{r-3} intersect the space in which is immersed the focal hyperboloid H_{r-2} in an $r-2$-dimensional sphere K_{r-2} whose equations are:

$$\sum_{n-r+1}^{n-2} X_i^2+\left(X_{n-1}+\frac{k}{a}\right)^2=\frac{k^2(1-a^2)-a^2}{a^2},\quad X_i=0,\quad i=1,2,\ldots,n-r,\quad (64)$$

which is tangent to H_{r-2} along Σ_{r-3}. In the same way, the point-spheres having their centres on Σ_{n-r-1}, intersect the space in which is immersed the focal ellipsoid E_{n-r} in a sphere K_{n-r} of $n-r$ dimensions whose equations are:

$$\sum_{1}^{n-r} X_i^2+(X_{n-1}+ka)^2=1-k^2(1-a^2),\quad X_i=0,\quad i=n-r+1,\ldots,n-2,\quad (65)$$

and this sphere is tangent to the focal ellipsoid E_{n-r} along Σ_{n-r-1}. The proof being very simple need not be given here. We shall call the spheres K_{r-2} and K_{n-r} *the focal loci of the cyclide.** *The focal loci are normal to the principal sphere* as may easily be proved from equations (63), (64) and (65).

We have seen that all the spheres S_e having their centres on the focal ellipsoid E_{n-r} pass through Σ_{r-3} and that all the spheres S_h having their centres on H_{r-2} pass through Σ_{n-r-1}. Consider now a cone with vertex at any point A on Σ_{r-3} and passing through the ellipsoid E_{n-r}. This cone is a cone of revolution whose axis is a tangent to the hyperboloid at A, which, as we have seen, passes through the centre C of the principal sphere. In fact, we prove that the radii of the spheres S_e which meet at A, make a constant angle with this tangent. The direction-cosines of any line PA, where P is any point on E_{n-r}, are:

$$\frac{X_1}{PA},\ \frac{X_2}{PA},\ \ldots,\ \frac{X_{n-r}}{PA},\ \frac{-\overline{X}_{n-r+1}}{PA}\cdots\frac{-\overline{X}_{n-2}}{PA},\ \frac{X_{n-1}-\overline{X}_{n-1}}{PA},$$

where $X_1, X_2, \ldots, X_{n-r} X_{n-1}$ are the coordinates of the point P on the ellipsoid, and $\overline{X}_{n-r+1}, \ldots, \overline{X}_{n-2}, \overline{X}_{n-1}$ those of a point on Σ_{n-3}. Again, the direction-cosines of the tangent to the hyperboloid at A are:

$$0,0,\ldots,0,\ \frac{\overline{X}_{n-r+1}}{CA},\ \ldots,\ \frac{\overline{X}_{n-2}}{CA},\ \frac{-a\left[k-\dfrac{1}{k(1-a^2)}\right]}{CA}.$$

*In adopting the above nomenclature we have followed the analogous one used by Darboux, *Leçons sur les systémes orthogonaux*, p. 489.

P14736

The distances $\overline{PA}$ and $\overline{CA}$ are:

$$PA = \frac{a\cos\theta_{n-r}}{\sqrt{1-a^2}} + k, \quad CA = \sqrt{\frac{[1-k^2(1-a^2)]\,[a^2-k(1-a^2)]}{k(1-a^2)}},$$

and the cosine of the angle between these two lines is found to be

$$\cos\omega = \frac{\sqrt{k^2(1-a^2)-1}}{\sqrt{k^2(1-a^2)-a^2}} = \text{const.}$$

In the same way we may prove that the cosine of the angle between the lines $\overline{P'B}$ and $\overline{CB}$ where P' is any point on H_{r-2} and B a point on Σ_{n-r-1}, is constant and equal to

$$\cos\omega' = \frac{\sqrt{a^2-k^2(1-a^2)}}{\sqrt{1-k^2(1-a^2)}} = \sec\omega = \text{const.}$$

and, since $\omega' = \frac{\pi}{2} - \alpha'$, where α' is the angle between the spheres S_h having its centre at P' and the focal locus K_{n-r}, we have

$$\sin\omega' = \cos\alpha' = \frac{i\sqrt{1-a^2}}{\sqrt{k^2(1-a^2)-1}}.$$

Hence, if ω is a real angle, ω' is imaginary and vice versa. We may now state the result in the following

THEOREM. *A general cyclide of Dupin of the fourth order in S_{n-1} is the envelope of spheres whose centres lie on an ellipsoid of revolution E_{n-r} and which intersect a fixed $r-2$-dimensional sphere K_{r-2} at a fixed angle α. It is also the envelope of spheres whose centres lie on a hyperboloid H_{r-2}, and which intersect an $n-r$-dimensional sphere K_{n-r} at a constant angle α'.*

It remains to discuss the nature of the loci of double points on the cyclide. For real cyclides the equation of the principal sphere

$$\sum_1^{n-2} X_i^2 + \left[X_{n-1} + \frac{a}{k(1-a^2)}\right]^2 = \frac{\left[\frac{1}{1-a^2} - \frac{a^2}{k^2(1-a^2)^2}\right]\left[\frac{a^2}{1-a^2} - \frac{a^2}{k^2(1-a^2)}\right]}{\frac{a^2}{k^2(1-a^2)}}$$

is also real. Let the centre on the X_{n-1}-axis $= \frac{a}{k(1-a^2)}$ be inside the ellipsoid and between the two sheets of the hyperboloid. Σ_{r-3} is then real, the focal locus K_{n-r} is then imaginary and K_{r-2} is real, as is also the principal sphere. The sphere Σ_{n-r} is imaginary and the cyclide has one real locus of double points only.

Let the centre C be between the foci of the hyperboloid and the ellipsoid. The principal sphere is imaginary, since the two factors in the numerator of R^2 have opposite signs. The focal spheres K_{r-2} and K_{n-r} are real since $k^2(1-a^2)-a^2$ is positive. The cyclide has no real double locus.

Finally, when the centre C is outside of the ellipsoid, the principal sphere is again real while Σ_{r-3} is imaginary. The sphere Σ_{n-r-1} is real and the cyclide has one real locus of double points.

§ 8. *Transformations.*

If we transform the cyclide by an inversion whose pole is on one of the focal loci, say K_{r-2}, it is transformed into a torus. In fact, the focal sphere K_{r-2} is transformed into an $r-2$-flat, and the spheres S_h which, as may easily be proved, intersects K_{r-2} orthogonally are transformed into spheres whose centres lie on this flat.* The transform of the cyclide is therefore the envelope of spheres whose centres lie on the $r-2$-flat and touch a given sphere, namely a transform of one of the spheres S_e. By revolving this latter sphere about the flat we get a torus, the transform of the cyclide. If the pole lies on the sphere K_{n-r} this sphere is transformed into an $n-r$-flat and the spheres S, which are orthogonal to K_{n-r}, are transformed into spheres whose centres lie on the $n-r$-flat. These spheres touch any one of the transforms of S_h and the new surface is therefore obtained by revolving the sphere S_h about the $n-r$-flat as an axis. The surface is therefore a torus. *There are ∞^{n-2} inversions which transform a cyclide into a torus.* If, in particular, the pole lies on the sphere Σ_{n-r-1} (or on Σ_{r-3}) we obtain a cone of revolution. There are therefore ∞^{n-r-1} inversions which will transform a cyclide of type $r\leq\frac{n+2}{2}$ into a cone of revolution.

Since the cyclide has only two focal sheets, the determinant in § 3, note, has only two roots, σ_1 and σ_2, of multiplicity $r-2$ and $n-r$, respectively. There exist, therefore, corresponding to $R_1\infty^{r-3}$ principal directions on the surface which lie in a space S_{n-r}; and corresponding to R_2 there are ∞^{n-r-1} principal directions lying in a space S_{r-2}.† Thus we found that from any point A on the locus of double points Σ_{r-3} pass ∞^{n-r-1} isotropic lines forming an isotropic cone of revolution whose elements are lines of curvature. Through any point P not on Σ_{r-3} or Σ_{n-r-1} pass two pencils of ∞^{n-r-1} and ∞^{r-3} circles, respectively. The circles through P generate two spherical spaces S_{n-r} and

---

* As a consequence we have a new mode of generation: *a cyclide of the fourth order is generated in two ways by spheres whose centres lie on a hyperboloid of revolution H_{r-2} (or an ellipsoid of revolution E_{n-r}) and which intersect a sphere K_{r-2} (or K_{n-r}) at right angles.*

† See Bianchi, *Lezioni*, Vol. I. p. 369. Seconda edizione.

S_{r-2} which intersect at right angles, as may easily be proved by transforming the cyclide into a cone of revolution.

We consider two special cases. If $k=0$, Σ_{r-3} is the imaginary sphere $\sum_{n-r+1}^{n-2} X_i^2=-1$, $X_1=\ldots=X_{n-r}=X_{n-1}=0$ and Σ_{n-r-1}' becomes the unit sphere $\sum_1^{n-r} X_i^2=1$, $X_{n-r+1}=X_{n-r+2}=\ldots=X_{n-1}=0$. The cyclide is symmetrical with respect to all the coordinate planes.

If Σ_{r-3} is the null-sphere whose centre is on the vertex of the focal hyperboloid we have $k=\dfrac{1}{\sqrt{1-a^2}}$. The focal sphere K_{n-r} becomes the point-sphere and Σ_{n-r-1} an imaginary sphere with radius $=\dfrac{\sqrt{a^2-1}}{a}$ which is immersed in the directrix-space of the focal ellipsoid.

An inversion with pole on Σ_{n-r-1} or Σ_{r-2} will transform the cyclide into a cylinder of revolution. The cyclide will have a single real double locus in finite space as in the general case.

§ 9. *Cyclides of the Third Order.*

If $a=1$ we have a type of cyclides of the third order, the parabolic type, obtained in § 6 (49′). The locus of point-spheres on the surface are gotten by equating to zero the radii of the two sets of spheres (50) which we shall denote by S_{p_1} and S_{p_2}. We have then

$$1+2\sum_1^{n-r}\rho_i^2=k,\quad 1+2\sum_{n-r+1}^{n-2}\rho_i^2=-k. \tag{66}$$

Substituting in equations (50) we have the two point-spheres

$$\left.\begin{aligned}&\sum_1^{n-r-1}(X_i+2\rho_i)^2+\left(X_{n-r}\pm 2\sqrt{\frac{k+1}{2}-\sum_1^{n-r-1}\rho_i^2}\right)^2+\sum_{n-r+1}^{n-2}X_i^2+\left(X_{n-1}+\frac{2-k}{2}\right)^2=0,\\ &\sum_1^{n-r}X_i^2+\sum_{n-r+1}^{n-3}(X_i-2\rho_i)^2+\left(X_{n-2}\mp 2\sqrt{\frac{-(1+k)}{2}-\sum_{n-r+1}^{n-2}\rho_i^2}\right)^2\\ &\qquad\qquad\qquad\qquad\qquad\qquad+\left(X_{n-1}-\frac{2+k}{2}\right)^2=0.\end{aligned}\right\} \tag{67}$$

The loci of point-spheres are therefore:

$$\left.\begin{aligned}&X_1=-2\rho_1,\ \ldots,\ X_{n-r-1}=-2\rho_{n-r-1},\quad X_{n-r}=\mp 2\sqrt{\frac{k+1}{2}-\sum_1^{n-r-1}\rho_i^2},\\ &\qquad X_{n-r+1}=\ldots=X_{n-2}=0,\quad X_{n-1}=\frac{k}{2}-1,\\ &X_1=\ldots=X_{n-r}=0,\quad X_{n-r+1}=2\rho_{n-r+1}=\ldots=X_{n-3}=2\rho_{n-3},\\ &\qquad X_{n-2}=\pm 2\sqrt{\frac{-(1+k)}{2}-\sum_{n-r+1}\rho_i^2},\quad X_{n-1}=1+\frac{k}{2}.\end{aligned}\right\} \tag{68}$$

The linear element of the surface is

$$ds^2 = \Sigma b_i d\theta_i,$$

where the b's have the values:

$$\left.\begin{aligned}
b_1 &= [\sin\theta_2 \dots \sin\theta_{n-r}\phi_{n-2} + \sin\theta_2 \dots \sin\theta_{n-r-1}\phi_{n-r} \\
&\qquad + \dots + \phi_2 + \Phi_1 + \Phi_1'']^2, \\
b_2 &= [\sin\theta_3 \dots \sin\theta_{n-r}\phi_{n-2} + \sin\theta_3 \dots \sin\theta_{n-r-1}\phi_{n-r} \\
&\qquad + \dots + \phi_3 + \Phi_2 + \Phi_2'']^2, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
b_{n-r} &= [\phi_{n-2} + \Phi_{n-r} + \Phi''_{n-r}]^2, \quad b_{n-r+1} = [\psi_{n-2} + \Phi_{n-r+1} + \Phi''_{n-r+1}]^2, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
b_{n-3} &= [\cos\theta_{n-4} \dots \cos\theta_{n-r+1}\psi_{n-2} + \cos\theta_{n-4} \dots \cos\theta_{n-r+2}\psi_{n-r+1} \\
&\qquad + \dots + \psi_{n-4} + \Phi_{n-3} + \Phi''_{n-3}]^2, \\
b_{n-2} &= [\Phi_{n-2} + \Phi''_{n-2}]^2,
\end{aligned}\right\} \quad (75)$$

from which it is seen that the lines $\theta_1 = c_1$, $\theta_2 = c_2$, ..., $\theta_{n-3} = c_{n-3}$ are geodetic.

The moulding surfaces (74) have certain geometric properties which we shall now discuss.* There are $n-3$ types of these surfaces obtained by giving to r the successive values 3, 4, ..., $n-1$; however, these types are not distinct, in fact, the types for which $r=k$ and $r=n-k+2$ will not yield essentially different surfaces as is seen from equations (72) and (74). If n is odd there are $\frac{n-3}{2}$ distinct types, and $\frac{n-2}{2}$ if n is even.

Consider the surface V_{n-r},

$$\left.\begin{aligned}
\xi_1 &= \sin\theta_1 \dots \sin\theta_{n-r-1}\phi_{n-r} + \dots + \sin\theta_1\phi_2 + \phi_1, \\
\xi_2 &= \cos\theta_1 \sin\theta_2 \dots \sin\theta_{n-r-1}\phi_{n-r} + \dots + \cos\theta_1\phi_2 + \psi_1, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
\xi_{n-r} &= \cos\theta_{n-r-1}\phi_{n-r} + \psi_{n-r-1}, \quad \xi_{n-1} = \psi_{n-r}, \quad \xi_{n-r+1} = \xi_{n-r+2} = \dots = \xi_{n-2} = 0,
\end{aligned}\right\} \quad (76)$$

immersed in the space $\xi_{n-r+1} = \dots = \xi_{n-2} = 0$; it is a moulding surface of type $r'=3$ in a space S_{n-r+1}. Consider also the cylindrical surface C_{n-r+1} obtained by constructing the normals to the space S_{n-r+1} at every point of V_{n-r}; the direction cosines of these normals to V_{n-r} lying in S_{n-r+1} are:

$$\sin\theta_1 \dots \sin\theta_{n-r}, \; \dots, \; \cos\theta_{n-r-1}\sin\theta_{n-r}, \; 0, 0, \dots, 0, \; \cos\theta_{n-r},$$

and these are also the direction cosines of a normal to C_{n-r+1} at any point of

---

* A special case of the surfaces (74) has been obtained by Umberto Sbrana in an article entitled "I sistemi ciclici nello spazio euclidio ad n dimensione," *Rendiconti del Circolo Matematico di Palermo*, Tomo, XIX, pp. 258–290. The surface is one of type $r=3$, the only distinct type that can exist in 4-space ($n=5$) since $\frac{n-3}{2}=1$ for $n=5$.

the generator passing through the point $\theta_1 \ldots \theta_{n-r}$. We now construct a surface V_{r-2}:

$$\left.\begin{array}{l} \zeta_{n-r+1} = \cos\theta_{n-3} \ldots \cos\theta_{n-r+1}\psi_{n-2} + \ldots + \cos\theta_{n-3}\psi_{n-4} + \psi_{n-3}, \\ \zeta_{n-r+2} = \sin\theta_{n-3} \ldots \cos\theta_{n-r+1}\psi_{n-2} + \ldots + \sin\theta_{n-3}\psi_{n-4} + \phi_{n-3}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \zeta_{n-2} \;\; = \sin\theta_{n-r+1}\psi_{n-2} + \phi_{n-r+1}, \quad \zeta_{n-1} = \phi_{n-2}, \quad \zeta_{n-r} = \ldots = \zeta_1 = 0, \end{array}\right\} \quad (77)$$

which is a moulding surface in space S_{r-1} of type $r'=3$. This space has the maximum of perpendicularity with the space S_{n-r+1}.

A moulding surface V_{n-2} in S_{n-1} may now be constructed geometrically as follows: In the space S_{n-r+1} of the ambient space S_{n-1} we construct a moulding surface V_{n-r} of type $r'=3$, and a cylindrical surface C_{n-r+1} which is the locus of normals to S_{n-r+1} erected at all points of V_{n-r}. We then construct a moulding surface V_{r-2} in space S_{r-1} which has the maximum of perpendicularity with S_{n-r+1} with a convenient orientation of axes $\zeta_{n-r+1}, \ldots, \zeta_{n-1}$. Let this space take all the ∞^{n-r} positions that are obtained by making a certain one of the given axes coincide with a generator of the cylinder C_{n-r+1}, and the remaining $r-2$-axes with the $r-2$ oriented directions normal to C_{n-r+1} in the point in which the generator meets V_{n-r}. The surface generated will be a moulding surface V_{n-2}. This generation may be described as a "rolling" of S_{r-1} on the cylinder C_{n-r+1}, the points of V_{r-2} in S_{r-1} generating the surface V_{n-2}. A moulding surface in S_{n-1} of type $r=3$ is thus generated by a plane S_2 rolling on a cylinder C_{n-2}, the points of the curve V_1 in the plane generating the surface. Hence it follows that *starting with a moulding surface in 3-space* (with cylindrical directrix) *the moulding surfaces of type $r'=3$ may be generated in the next higher spaces up to, say, S_{n-r+1} and S_{r-1}, and finally V_{n-2} by "rolling" the space S_{r-1} on the cylinder constructed with V_{n-r} as a base. S_{r-1} must have maximum perpendicularity with S_{n-r+1}. The surface generated by the points of V_{r-2} is of type r.*

If in (74) we put $\theta_{n-2}=c_{n-2}$ we get two moulding surfaces of type $r'=3$, one immersed in a space $S(X_1, X_2, \ldots, X_{n-r}, X_{n-1})$, and the other in a space $S(X_{n-r+1}, \ldots, X_{n-2})$, both being curved varieties on V_{n-2}. Starting with a moulding surface in 3-space we may construct each one of these by rolling a plane in which is drawn a profile-curve $\xi_i=\phi_i$, $\eta_i=\Psi_i$, over a cylindrical directrix C_i whose base is a moulding surface V_{i-1} of type $r'=3$ in the next lower space; since the profile-curve is plane it will generate a set of plane lines of curvature on V_i. The lines of curvature on V_i remain lines of curvature on V_{i+1} in the next higher space. But the lines of curvature θ_{n-2} are also plane on V_{n-2}, hence they are all plane.

The $n-2$ focal sheets are all distinct; in fact, if we calculate the values of the principal radii we find:

$$R_1=\frac{-\sqrt{b_1}}{\sin\theta_2\ldots\sin\theta_{n-r}\sin\theta_{n-2}},\ \ldots,\quad R_{n-r}=\frac{-\sqrt{b_{n-r}}}{\sin\theta_{n-2}},$$

$$R_{n-r+1}=\frac{-\sqrt{b_{n-r+1}}}{\cos\theta_{n-2}},\ \ldots,\quad R_{n-3}=\frac{-\sqrt{b_{n-3}}}{\cos\theta_{n-r+1}\ldots\cos\theta_{n-4}\cos\theta_{n-2}},$$

$$R_{n-2}=-\sqrt{b_{n-2}}.$$

§ 11. *Surfaces of Revolution.*

If $\Phi_1=\Phi_2=\ldots=\Phi_{n-r}=0$, we get a species of surfaces of revolution in S_{n-1}. If we introduce the arc of the curve $\zeta_{n-2}=\psi_{n-2}$, $\zeta_{n-1}=\phi_{n-2}$ as a new parameter u instead of θ_{n-2}, the equations of the surface may be written

$$X_1=\sin\theta_1\ldots\sin\theta_{n-r}U,\ \ldots,\quad X_{n-r}=\cos\theta_{n-r+1}\sin\theta_{n-r}U,$$

$$X_{n-r+1}=\cos\theta_{n-3}\ldots\cos\theta_{n-r+1}\int_{u_0}^{u}\sqrt{1-U'^2}+\cos\theta_{n-3}\ldots\cos\theta_{n-r+2}\psi_{n-r+1}$$
$$+\ldots+\cos\theta_{n-3}\psi_{n-4}+\psi_{n-3},$$

$$X_{n-r+2}=\sin\theta_{n-3}\ldots\cos\theta_{n-r+1}\int_{u_0}^{u}\sqrt{1-U'^2}+\sin\theta_{n-3}\ldots\cos\theta_{n-r+2}\psi_{n-r+1}$$
$$+\ldots+\sin\theta_{n-3}\psi_{n-4}+\phi_{n-3},$$

. . . . ,

$$X_{n-2}=\sin\theta_{n-r+1}\int_{u_0}^{u_1}\sqrt{1-U'^2}+\phi_{n-r+1},\quad X_{n-1}=\cos\theta_{n-r}U,$$

generated by the moulding surface V (equations 76), turning about the coordinate flat $X_1=X_2=\ldots=X_{n-r}=X_{n-1}=0$ as an axis. Consider the variety $\theta_{n-r+1}=c_{n-r+1}$, ..., $\theta_{n-2}=c_{n-3}$, $u=c$ on the surface; it is a sphere of constant radius $U(c)$ and its centre lies on the axial space $X_1=\ldots=X_{n-r}=X_{n-1}=0$. Let c be fixed while θ_{n-r+1}, ..., θ_{n-3} vary; the locus of centres is a moulding surface in $S(X_{n-r+1},\ldots,X_{n-2})$ of type $r'=3$. Any line traced on the ∞^{r-2} spheres $u=c$, where c has any value, is a line of curvature on the surface. Since $R_1=R_1=\ldots=R_{n-r}$ the surface has $r-1$ distinct focal sheets. A surface of revolution of type $r=3$ has $r-1=2$ distinct focal sheets.*

§ 12. *Transformations of the Gaussian Sphere. Orthogonal Systems.*

From equations (37) it follows that given the spherical representation (29) the function $W=\Sigma\Phi_i\sqrt{E_i}$ defines all surfaces whose lines of curvature are plane in all systems. The function W satisfies the following system of differential equations:

$$\frac{\partial^2 W}{\partial\theta_i\partial\theta_k}=\frac{\partial\log\sqrt{E_k}}{\partial\theta_i}\frac{\partial W}{\partial\theta_k}+\frac{\partial\log\sqrt{E_i}}{\partial\theta_k}\frac{\partial W}{\partial\theta_i};\qquad i,k=1,2,\ldots,n-2,\quad i\neq k.\quad (78)$$

*A surface of this kind ($r=3$) in 4-space has been constructed by Sbrana, *loc. cit.*, p. 287.

If now we put

a) $$\lambda=\Delta_1'W+(\theta_{n-1}+W)^2=\sum\frac{1}{E_i}\left(\frac{\partial W}{\partial\theta_i}\right)^2+(\theta_{n-1}+W)^2,$$

where θ_{n-1} denotes an arbitrary constant, we have the following application of a general theorem proved by Sbrana:*

There exists a transformation of the Gaussian sphere into itself by means of which the differential form (30) *is transformed into one of the form*

$$d\sigma^2=\lambda^2\sum_1^{n-2}\left[\frac{\dfrac{\partial\left(\dfrac{W+\theta_{n-1}}{\lambda}\right)}{\partial\theta_i}}{\dfrac{\partial W}{\partial\theta_i}}\right]^2 E_i d\theta_i^2,\quad (\theta_{n-1}=\text{arbitrary constant.})$$

To a point x_i corresponds a point x_i' by means of the equations

b) $$x_i'=x_i-\rho\frac{\theta_{n-1}+W}{\Psi}\,[(\theta_{n-1}+W)x_i+\rho\nabla'(x_i,W)],$$

where

c) $$\Psi=\int\sum R_i\frac{\partial W}{\partial\theta_i}\,d\theta_i \text{ and } -\frac{2\Psi}{\rho}+\Delta'W+(\theta_{n-1}+W)^2=0.$$

The corresponding $n-1$-tuple orthogonal system has for linear element

$$ds^2=\Sigma d\xi_i^2=\rho^2 d\theta_{n-1}^2+\lambda^2\sum_1^{n-2}\left[\frac{\dfrac{\partial\left(\dfrac{W+\theta_{n-1}}{\lambda}\right)}{\partial\theta_i}}{\dfrac{\partial W}{\partial\theta_i}}\right]^2 E_i d\theta_i^2,$$

the coordinates ξ_i being given by the equations:

$$\xi_i=X_i+\rho(\theta_{n-1}+W)x+\rho\nabla'(x_i,W),$$

the symbol ∇' in (b) *being equivalent to the summation*

$$\sum\frac{1}{E_i}\frac{\partial x_i'}{\partial\theta_i}\frac{\partial W}{\partial\theta_i}.$$

All the cyclic systems normal to the surfaces whose tangential equation is given by (37) may be obtained by quadrature. In particular, the cyclic systems normal to the surfaces (74) are gotten by integrating the equation

$$d\Psi=\sum_1^{n-2}R_i\frac{\partial W}{\partial\theta_i}\,d\theta_i,$$

* Sbrana, *loc. cit.*, p. 227. An application to a very special case is given on p. 282.

where the R's have the values given at the end of §10 and $W=\Sigma\Phi_i\sqrt{E_i}$, (eq. 71). For the general surface (37) the calculation of the R's involves considerable work.

II.

§1. *Transformations. Asymptotic Lines.*

Consider the $n-2$-spread (6),

$$\left.\begin{aligned} X_1 &= \frac{(\alpha_1+\beta_1)}{2} X_{n-1} - \frac{1}{2}(F_{\alpha_1}+F_{\beta_1}), \\ X_2 &= i\frac{(\alpha_1-\beta_1)}{2} X_{n-1} + \frac{i}{2}(F_{\alpha_1}-F_{\beta_1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-1} &= \frac{\Sigma\alpha_i F'_{\alpha_i}+\Sigma\beta_i F'_{\beta_i}-F}{1+\Sigma\alpha_i\beta_i}. \end{aligned}\right\} \tag{79}$$

If, as was done in our former paper (A, p. 203) we consider F, α_i, β_i, F'_{α_i}, F'_{β_i} as the surface-elements of a space S_{n-1}, the above system may be looked upon as a contact-transformation which carries the surface-elements of S_{n-1} into those of $\overline{S}_{n-1}$. To the lines of curvature on a surface in S_{n-1} correspond a set of conjugate lines on the transform to which we have given the name Euler's lines or E-lines. These lines have the property of being transformed into asymptotic lines on a surface in a space $\overline{\overline{S}}_{n-1}$ by means of Euler's transformation

$$x_i=F'_{\alpha_i}, \quad y_i=\beta_i, \quad p_i=-\alpha_i, \quad q_i=F'_{\beta_i}, \quad z=F-\Sigma\alpha_i F'_{\alpha_i}. \tag{80}$$

These E-lines, while they have been implicitly used by several authors,* have never been considered from the view-point of the theory of contact-transformations before the author's treatment of them in an article in *American Transactions*, Vol. VI, pp. 450–471. Since Euler's transformation applies to odd space the E-lines do not exist on a surface in even-dimensional space, and, as a consequence, the asymptotic lines exist in odd space only.

The tangential equation of a surface in an even or odd space S_{n-1} may be written as before:

$$2y_1X_1+2y_2X_2+\ldots\ldots+(1-\Sigma y_i^2)X_{n-1}+F=0, \tag{I}$$

and, if S_{n-1} is an odd space,

$$(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\ldots\ldots+(1-\Sigma\alpha_i\beta_i)X_{n-1}+F=0, \tag{II}$$

* Darboux, *Leçons*, Vol. I, pp. 200–202, deuxième édition, Vol. IV, p. 171.

which latter form is reducible to the first by means of the transformation

$$\alpha_1+\beta_1=2y_1, \quad i(\alpha_1-\beta_1)=2y_2, \ldots, \quad (1-\Sigma\alpha_i\beta_i)=1-\Sigma y_i^2.$$

The lines of curvature of the surface M_{n-2} defined as the envelope of the tangent planes (II) are given by the system of differential equations (A, p. 211),

$$\frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\alpha k}}{dF_{\beta_{\frac{n-2}{2}}}}, \quad \frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}}=\frac{dF'_{\beta i}}{dF_{\beta_{\frac{n-2}{2}}}}, \qquad \begin{array}{l} i=1, 2, \ldots, \frac{n-4}{2}, \\ k=1, 2, \ldots, \frac{n-2}{2}, \end{array}$$

to which in the case of (I) corresponds the system

$$\frac{dy_i}{dy_{n-2}}=\frac{dF'_{y_i}}{dF_{n-2}}, \qquad i=1, 2, \ldots, n-3, \tag{81}$$

and we have seen (§ 2) that if the lines of curvature on a surface are *coordinate lines,* the y's, considered as functions of the ρ's, must form a completely orthogonal system in an $n-2$-space, and that F is a particular or general solution of the system of differential equations (12). The contact-transformation corresponding to (6) now takes the simple form:

$$X_i=\frac{y_i(\Sigma y_iF'_{y_i}-F)}{1+\Sigma y_i^2}-\tfrac{1}{2}F'_{y_i}, \quad X_{n-1}=\frac{\Sigma y_iF'_{y_i}-F}{1+\Sigma y_i^2}, \tag{82}$$

which transforms spheres into paraboloids of the form:

$$F=a\Sigma y_i^2+\Sigma b_iy_i+c.$$

The Theorem II, (§ 2), is therefore true in any space, and the surfaces obtained by the application of the theory exist in any space, even or odd.

If we consider an odd space $\bar{S}_{n-1}$ and a spread $F=F(\alpha_i, \beta_i)$, Theorem II may be stated thus:

If, on a surface in $\bar{S}_{n-1}$, a system of curves are coordinate E-lines, the coordinates F, α_i, β_i satisfy the system of differential equations

$$(\lambda_k-\lambda_{k'})\frac{\partial^2\theta}{\partial\rho_k\partial\rho_{k'}}-\frac{\partial\lambda_{k'}}{\partial\rho_k}\frac{\partial\theta}{\partial\rho_{k'}}-\frac{\partial\lambda_k}{\partial\rho_{k'}}\frac{\partial\theta}{\partial\rho_k}=0, \tag{12a}$$

and α_i, β_i satisfy the $\frac{n-2\cdot n-3}{2}$ relations

$$\sum\left(\frac{\partial\alpha_i}{\partial\rho_{k'}}\frac{\partial\beta_i}{\partial\rho_k}+\frac{\partial\alpha_i}{\partial\rho_k}\frac{\partial\beta_i}{\partial\rho_{k'}}\right)=0, \qquad k, k'=1, 2, \ldots, n-2,$$

and the surface F is found by taking any particular or general solution of (12a).

If we transform to $\bar{\bar{S}}_{n-1}$, using Euler's transformation, we have the following

THEOREM III. *If on a surface in $\overline{\overline{S}}_{n-1}$ a system of curves are coordinate asymptotic lines, the coordinates y_i and p_i of the surface-elements on the surface must satisfy the differential equations* (12a) *and also the* $\frac{n-2 \cdot n-3}{2}$ *relations*

$$\sum\left(\frac{\partial y_i}{\partial \rho_{k'}}\frac{\partial p_i}{\partial \rho_k}+\frac{\partial y_i}{\partial \rho_k}\frac{\partial p_i}{\partial \rho_{k'}}\right)=0.$$

This theorem solves the problem of *Lelieuvre* for general odd space. A system of solutions y_i, p_i having been found, q_i and x_i may be found by quadratures and hence also z. It should be observed that $z-\Sigma x_i p_i$ is a particular solution of (12a).

In order to find surfaces with coordinate asymptotic lines it will be most convenient to find in S_{n-1} corresponding surfaces with coordinate lines of curvature. The chief difficulty which we encounter is the fact, pointed out before, that to real elements in S_{n-1} correspond in general imaginary elements in $\overline{\overline{S}}_{n-1}$ with the obvious result that the surface, if real in S_{n-1}, has often an imaginary transform in $\overline{\overline{S}}_{n-1}$ and vice versa. If, however, we have a real surface in $\overline{S}_{n-1}$, *i. e.*, if F considered as a function of α_i, β_i, is real, the corresponding surface in $\overline{\overline{S}}_{n-1}$ is real.

§ 2. Consider the surfaces for which

$$F=2\phi_1(\rho_1)+2\phi_2(\rho_2)+\ldots+2\phi_{n-2}(\rho_{n-2})+2C$$

obtained on p. 7. If we transform by Euler's transformation we get the surface

$$\left.\begin{aligned} x_i&=\phi'_{2i-1}+i\phi'_{2i}, \quad y_i=\rho_{2i-1}+i\rho_{2i}, \\ z&=2\Sigma\phi_i(\rho_i)-\Sigma(\rho_{2i-1}-i\rho_{2i})(\phi'_{2i-1}+i\phi'_{2i})+2C, \end{aligned}\right\} \quad (83)$$

which is real if ϕ_{2i} is an even function of ρ_{2i}.

As a particular case let us take a cyclide of the third order and type $r=\frac{n-2}{2}$ obtained by giving to the ϕ's the following values: *

$$\phi_{2i-1}=\frac{k+2}{2}\rho^2_{2i-1}, \quad \phi_{2i}=\frac{k-2}{2}\rho^2_{2i}, \quad 2C=\frac{k}{2}.$$

We get the real surface

$$x_i=\frac{k+2}{2}\rho_{2i-1}+i\frac{k-2}{2}\rho_{2i}, \quad y_i=\rho_{2i-1}+i\rho_{2i}, \quad z=2i\Sigma\rho_{2i-1}\rho_{2i}+\frac{k}{2}, \qquad (84)$$

that is, the paraboloid

$$z+\frac{1}{2}\sum_1^{\frac{n-2}{2}}\left[x_i^2+\frac{k^2-4}{4}y_i^2-kx_iy_i\right]=\frac{k}{2}. \qquad (84')$$

* This surface differs from (49′) only in orientation of axes.

This surface is generated in two different ways by self-dual flats corresponding to the two sets of curvature-spheres which generate its transform in S_{n-1}. These flats are evidently:

$$\left.\begin{aligned} x_i &= \frac{k-2}{2} y_i + 2\rho_{2i-1}, & x_i &= \frac{k+2}{2} y_i - 2i\rho_{2i}, \\ z &= 2\Sigma[\rho_{2i-1}y_i - \rho^2_{2i-1}] + \frac{k}{2}, & z &= 2i\Sigma[\rho_{2i}y_i - i\rho^2_{2i}] + \frac{k}{2}. \end{aligned}\right\} \tag{85}$$

If we introduce the parameters α_i, β_i instead of the ρ's in these equations we see that they are real flats. The asymptotic lines are to a certain extent indeterminate. In fact, the flats I and II lie entirely in the surface (84′), and through any point on the surface pass two flats so that any curve immersed in either one of them is an asymptotic curve. Through any point will pass $\infty^{\frac{n-4}{2}}$ asymptotic directions.

To the flat
$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d, \tag{86}$$
corresponds a sphere (A, p. 217)

$$\left(X_1 + \frac{b_1+c_1}{2}\right)^2 + \left(X_2 - i\frac{b_1-c_1}{2}\right)^2 + \dots + \left(X_{n-1} - \frac{a-d}{2}\right)^2 = \left(\frac{a+d}{2}\right)^2,$$

but to a sphere corresponds two flats since $R = \pm\frac{a+d}{2}$. We shall call the flat for which $R = +\frac{a+d}{2}$ the *positive correspondent* and the one for which $R = -\frac{a+d}{2}$ the *negative correspondent.* To point-spheres correspond flats for which $a = -d$, that is, the two correspondents coincide. Any pair of + and — correspondents in the same set are said to be *conjugate* with respect to the flat-complex $a+d=0$ (A, p. 217). Comparing now equations (51), p. 20, with (85) and (86), it appears that the flats of the first set are negative correspondents, and those of the second set are positive correspondents. To the nodal loci on the cyclide correspond two sets of flats on the paraboloid (84) belonging to the flat-complexes $a+d=0=\frac{k-2}{2} - 2\Sigma\rho^2_{2i-1} + \frac{k}{2}$, and $a+d=0 = \frac{k+2}{2} + 2\Sigma\rho^2_{2i} + \frac{k}{2}$:

$$\left.\begin{aligned} x_i &= \frac{k-2}{2} y_i + 2\rho_{2i-1}, & x &= \frac{k+2}{2} y_i - 2i\rho_{2i}, \\ z &= 2\Sigma\rho_{2i-1}y_i - \frac{k}{2} + 1, & z &= 2\Sigma\rho_{2i}y_i - \frac{k}{2} - 1, \\ 2\Sigma\rho^2_{2i-1} &= (k-1), & 2\Sigma\rho^2_{2i} &= -(1+k). \end{aligned}\right\} \tag{87}$$

They are the common generators of the quadric and the two cylinders,

$$\Sigma\left(x_i-\frac{k-2}{2}\,y_i\right)^2=2(k-1),\quad \Sigma\left(x_i-\frac{k+2}{2}\,y_i\right)^2=-2(1+k). \tag{88}$$

If the cyclide is of type $r\neq\frac{n-2}{2}$, we give to the ϕ's the following values:

$$\phi_{2i-1}=\frac{k+2}{4}\rho^2_{2i-1},\quad \phi_{2i}=\frac{k-2}{4}\rho^2_{2i},\quad \phi_{2r+t}=\frac{k-2}{4}\rho^2_{2r+t},\qquad \begin{matrix} i=1,2,\ldots,r,\\ t=1,2,\ldots,r.\end{matrix}$$

The transform of the cyclide is:

$$\left.\begin{aligned} x_j&=\frac{k+2}{2}\rho_{2j-1}+i\frac{k-2}{2}\rho_{2j}, & y_i&=\rho_{2i-1}+i\rho_{2i}, \quad \begin{matrix} j=1,2,\ldots,r,\\ i=1,2,\ldots,\frac{n-2}{2},\end{matrix}\\ x_{r+s}&=\frac{k-2}{2}\left[\rho_{2(r+s)-1}+i\rho_{2(r+s)}\right]y_{r+s}, & z&=i\Sigma\rho_{2j}\rho_{2j-1}+\frac{k}{2},\\ & & s&=1,2,\ldots,\frac{n-2-2r}{2}.\end{aligned}\right\} \tag{89}$$

Eliminating the ρ's we get the parabolic surface of $\frac{n-2}{2}+r$ dimensions

$$z=-\frac{1}{2}\sum_1^r\left[x_j^2+\frac{k^2-4}{4}y_j^2-kx_jy_j\right]+\frac{k}{2};\quad x_{r+s}=\frac{k-2}{2}y_{r+s},$$
$$s=1,2,\ldots,\frac{n-2}{2}-r. \tag{90}$$

As in the preceding special case the two sets of spheres generating the cyclide are transformed into two sets of flats:

$$\mathrm{I}\begin{cases} x_j=\dfrac{k-2}{2}\,y_i+2\rho_{2j-1},\\ x_{r+s}=\dfrac{k-2}{2}\,y_{r+s},\\ z=2\sum\limits_1^r\rho_{2j-1}y_j-2\sum\limits_1^r\rho^2_{2j-1}+\dfrac{k}{2}.\end{cases}$$

$$\mathrm{II}\begin{cases} x_j=\dfrac{k+2}{2}\,y_j-2i\rho_{2j},\\ x_{r+s}=\dfrac{k+2}{2}\,y_{r+s}-2\left(\rho_{2(r+s)-1}+i\rho_{2(r+s)}\right),\\ z=2i\Sigma\rho_{2j}y_j-2\Sigma\left(\rho_{2(r+s)-1}-i\rho_{2(r+s)}\right)y_{r+s}+2\sum\limits_1^{\frac{n-2}{2}}\rho^2_{2j}+2\sum\limits_{r+1}^{\frac{n-2}{2}}\rho^2_{2j-1}+\dfrac{k}{2},\\ \qquad\left(j=1,2,\ldots,r,\quad s=r+1,\ldots,\dfrac{n-2}{2}-r\right).\end{cases}$$

The flats of the first set lie entirely in the surface and may be considered as generating the surface, the Cartesian equation of which is obtained by eliminating the parameters ρ_{2j}. The flats of the second set intersect those of the first in points on the surface, hence the parametric equation (89) may be gotten by solving I and II for x_i, y_i, and z.

Since to every sphere of radius $R \neq 0$ correspond two flats which do not intersect, to every surface in S_{n-1} correspond two surfaces in $\overline{\overline{S}}_{n-1}$; these are said to be conjugate to each other.* The conjugate of the surface (84) is obtained by changing the sign of R_1 and R_2 so that we have

$$\frac{a+d}{2}=R_1=\frac{1}{2}\,[2\Sigma\rho^2_{2i-1}+1-k],\quad R_2=-\frac{1}{2}\,[2\Sigma\rho^2_{2i}+1+k],$$
$$\frac{a-d}{2}=-\frac{1-2\Sigma\rho^2_{2i-1}}{2},\quad \frac{a-d}{2}=\frac{1-2\Sigma\rho^2_{2i}}{2},$$

from which we have

$$a=2\Sigma\rho^2_{2i-1}-\frac{k}{2},\quad d=1-\frac{k}{2}\,;\quad a=-2\Sigma\rho^2_{2i}-\frac{k}{2},\quad d=-\frac{k+2}{2},$$

the conjugate surface is therefore generated by either one of the two sets of flats

$$\left.\begin{array}{ll} x_i=\left(2\Sigma\rho^2_{2i-1}-\dfrac{k}{2}\right)y_i+2\rho_{2i-1}, & x_i=-\left(2\Sigma\rho^2_{2i}+\dfrac{k}{2}\right)y_i-2i\rho_{2i}, \\ z=2\Sigma\rho_{2i-1}y_i+1-\dfrac{k}{2}, & z=2i\Sigma\rho_{2i}y_i-\dfrac{k+2}{2}. \end{array}\right\} \quad (91)$$

Eliminating the ρ's we have the quartic surface

$$\sum_1^{\frac{n-2}{2}}(x_iy_k-x_ky_i)^2-2\sum_1^{\frac{n-2}{2}}x_iy_i-k\Sigma y_i^2+z(z+k)+\frac{k^2-4}{4}=0,\qquad (i<k),\quad (84'')$$

which intersects the quadric (84′), *i. e.*, its conjugate, along the null-flats of the first and second sets, that is, flats for which

$$2\Sigma\rho^2_{2i-1}+1-k=0 \text{ and } 2\Sigma\rho^2_{2i}+1+k=0,$$

(equations (87) and (88)).

If the cyclide is of type $r\neq\dfrac{n-2}{2}$ the equations of the surface conjugate to (90) are

$$\left.\begin{array}{l} \sum\limits_1^r(x_iy_k-x_ky_i)^2-2\sum\limits_1^r x_iy_i-k\sum\limits_1^r y_i^2+z(z+k)+\dfrac{k^2-4}{4}=0, \\ \sum\limits_1^r(y_ix_{r+s}-x_ky_{r+s})y_i+zy_{r+s}+\dfrac{k-2}{2}y_{r+s}=0,\quad s=1,2,\ldots,\dfrac{n-2}{2}-r. \end{array}\right\}\ (90')$$

*Also "reciprocal" in the terminology of S. Lie.

Since the flats of the second set do not lie on the surface (90) the null-flats of the second set will not lie on the intersection of (90) and (90′); only if $r=\frac{n-2}{2}$ will this be true.

§ 3. *Transformation of the Cyclides of the Fourth Order.*

We write the two sets of spheres which generate the cyclide of type $r=\frac{n+2}{2}$ as follows:*

$$\left.\begin{aligned}&\Sigma X_i^2+2\cot\theta_{n-2}\cos\theta_{\frac{n}{2}}\ldots\cos\theta_{n-3}X_2+\ldots+2\cot\theta_{n-2}\sin\theta_{\frac{n}{2}}\cdot X_{n-2}\\&\qquad+\frac{2a_0\csc\theta_{n-2}}{\sqrt{1-a_0^2}}X_{n-1}=1+k^2-\frac{2k}{\sqrt{1-a_0^2}}\csc\theta_{n-2},\\&\Sigma X_i^2-2\sin\theta_{\frac{n-2}{2}}\ldots\sin\theta_1X_1-\ldots-2\sin\theta_{\frac{n-2}{2}}\cos\theta_{\frac{n-4}{2}}X_{n-3}\\&\qquad-\frac{2\cos\theta_{\frac{n-2}{2}}}{\sqrt{1-a_0^2}}X_{n-1}=1+k^2-2\left[1-\frac{a_0k}{\sqrt{1-a_0^2}}\cos\theta_{\frac{n-2}{2}}\right].\end{aligned}\right\}\quad(92)$$

Transforming we get two sets of flats,

$$\mathrm{I(a)}\begin{cases}x_i=ay_i+b_i,\\ z=-\sum_1^{\frac{n-2}{2}}b_iy_i+d,\end{cases}\qquad \mathrm{II(a)}\begin{cases}x_i=\bar a y_i+\bar b_i,\\ z=\sum_1^{\frac{n-2}{2}}\bar b_iy_i+\bar d,\end{cases}$$

where a, b_i, d, $\bar a$, $\bar b_i$, $\bar d$ have the following values:

$$a=-\sqrt{\frac{1+a_0}{1-a_0}}\csc\theta_{n-2}+k,\qquad \bar a=\sqrt{\frac{1+a_0}{1-a_0}}\cos\theta_{\frac{n-2}{2}}+k,$$

$$d=-\sqrt{\frac{1-a_0}{1+a_0}}\csc\theta_{n-2}+k,\qquad \bar d=-\sqrt{\frac{1-a_0}{1+a_0}}\cos\theta_{\frac{n-2}{2}}+k,$$

$$b_i=\ i\cot\theta_{n-2}\phi_i,\qquad \bar b_i=-\sin\theta_{\frac{n-2}{2}}\psi_i,$$

ϕ_i and ψ_i having the values:

$$\phi_1=\cos\theta_{\frac{n}{2}}\ldots\cos\theta_{n-4}\cos\theta_{n-3},\quad \phi_{i+1}=\cos\theta_{\frac{n}{2}}\ldots\cos\theta_{n-(i+3)}\sin\theta_{n-(i+2)},$$

$$\psi_1=\sin\theta_{\frac{n-4}{2}}\ldots\sin\theta_1,\qquad \psi_{i+1}=\sin\theta_{\frac{n-4}{2}}\ldots\sin\theta_i\cos\theta_{i-1},$$

$$\left(i=1,2,\ldots,\frac{n-4}{2}\right).$$

---

* These equations differ from (46), only in orientation of the axis $X_1,\ldots,X_{n-2}$; we also have put $a=a_0$ in order to avoid confusing it with the parameter a in I(a) and II(a).

Eliminating the θ's from I (or II) we obtain the quartic surface:

$$(1+a_0)\sum_1^{\frac{n-2}{2}}(x_iy_k-x_ky_i)^2+(1-a_0)\sum_1^{\frac{n-2}{2}}(x_i-ky_i)^2-(1+a_0)\sum_1^{\frac{n-2}{2}}y_i^2$$
$$+(1+a_0)(z-k)-1+a_0=0,\quad i<k. \quad (93)$$

The conjugate surface is:

$$(1-a_0)\sum_1^{\frac{n-2}{2}}(x_iy_k-x_ky_i)^2+(1+a_0)\sum_1^{\frac{n-2}{2}}(x_i+ky_i)^2-(1-a_0)\sum_1^{\frac{n-2}{2}}y_i^2$$
$$+(1-a_0)(z+k)^2-(1+a_0)=0, \quad (93')$$

generated by the two sets of flats:

$$\mathrm{I(b)}\ \begin{cases} x= & a'y_i+b_i, \\ z= & -\Sigma b_iy_i+d', \end{cases} \qquad \mathrm{II(b)}\ \begin{cases} x_i=\bar{a}'y_i+\bar{b}_i, \\ z\ =\Sigma\bar{b}_iy_i+\bar{d}', \end{cases}$$

where a', d', $\bar{a}'$, $\bar{d}'$ have the values

$$a'=\sqrt{\frac{1-a_0}{1+a_0}}\csc\theta_{n-2}-k, \qquad \bar{a}'=\sqrt{\frac{1-a_0}{1+a_0}}\cos\theta_{\frac{n-2}{2}}-k,$$
$$d'=\sqrt{\frac{1+a_0}{1-a_0}}\csc\theta_{n-2}-k, \qquad \bar{d}'=-\sqrt{\frac{1+a_0}{1-a_0}}\cos\theta_{\frac{n-2}{2}}-k.$$

These two surfaces intersect along the two sets of null-flats whose equations are:

$$x_i=-a_0ky_i+i\sqrt{1+k^2(1-a_0^2)}\,\phi_i, \qquad x_i=-\frac{k}{a_0}y_i-\frac{\sqrt{a^2-k^2(1-a_0^2)}}{a_0}\psi_i,$$
$$z=-\sum_1^{\frac{n-2}{2}}i\sqrt{1+k^2(1-a_0^2)}\,\phi_iy_i+ak, \qquad z=-\sum\frac{\sqrt{a_0^2-k^2(1-a_0^2)}}{a_0}\psi_iy_i+\frac{k}{a_0}.$$

These flats are the intersections of the quadric cylinders

$$\Sigma(x_i+a_0ky_i)^2=-[1+k^2(1-a_0^2)],\ \text{and}\ \Sigma\left(x_i+\frac{k}{a_0}y_i\right)^2=\frac{a_0^2+k^2(1-a_0^2)}{a_0^2}$$

with the surface (93) or (93′).

If the cyclide is of type $r<\frac{n+2}{2}$ we write the two sets of generating spheres as follows:

$$\left.\begin{array}{l}
\Sigma X_i^2+2\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3}X_2+\ldots \\
\quad +2\cot\theta_{n-2}\sin\theta_{n-r+1}X_{2(r-2)}+\dfrac{2a_0\csc\theta_{n-2}}{\sqrt{1-a_0^2}}X_{n-1}=1+k^2 \\
\quad -\dfrac{2k}{\sqrt{1-a_0^2}}\csc\theta_{n-2}, \\
\Sigma X_i^2-2\sin\theta_{n-r}\ldots\sin\theta_1X_1-2\sin\theta_{n-r}\ldots\cos\theta_1X_3-\ldots \\
\quad -2\sin\theta_{n-r}\ldots\cos\theta_{r-3}X_{2(r-2)-1}-2\sin\theta_{n-r}\ldots\cos\theta_{r-2}X_{2(r-2)+1} \\
\quad -\ldots-2\sin\theta_{n-r}\cos\theta_{n-r-1}X_{n-3}-\dfrac{2\cos\theta_{n-r}}{\sqrt{1-a_0^2}}X_{n-1} \\
\quad =1+k^2-2\left[1-\dfrac{a_0k}{\sqrt{1-a_0^2}}\cos\theta_{n-r}\right].
\end{array}\right\} \quad (94)$$

The parameters b_i, c_i, a, d, $\bar{b}_i$, $\bar{c}_i$, $\bar{a}$, $\bar{d}$ of the corresponding flats are therefore:

$$\left.\begin{aligned}
&b_1 = i\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\cos\theta_{n-3},\\
&b_2 = i\cot\theta_{n-2}\cos\theta_{n-r+1}\ldots\sin\theta_{n-3},\ \ldots,\\
&b_{r-2} = i\cot\theta_{n-2}\sin\theta_{n-r+1},\\
&a = -\sqrt{\frac{1+a_0}{1-a_0}}\csc\theta_{n-2}+k,\quad c_i=-b_i,\qquad i=1,2,\ldots,r-2,\\
&d = -\sqrt{\frac{1-a_0}{1+a_0}}\csc\theta_{n-2}+k,\quad b_{r-2+s}=c_{r-2+s}=0,\\
&\qquad\qquad s=1,2,\ldots,\frac{n+2}{2}-r,\\
&\bar{b}_1 = -\sin\theta_{n-r}\ldots\sin\theta_1,\quad \bar{b}_2=\sin\theta_{n-r}\ldots\cos\theta_1,\ \ldots,\\
&\bar{b}_{r-2} = -\sin\theta_{n-r}\ldots\cos\theta_{r-3},\quad \bar{c}_i=\bar{b}_i,\qquad i=1,2,\ldots,r-2,\\
&\bar{b}_{r-2+s} = -\sin\theta_{n-r}\ldots\sin\theta_{r-2+2s}(\sin\theta_{r-3+2s}\cos\theta_{r-4+2s}+i\cos\theta_{r-3+2s}),\\
&\bar{c}_{r-2+s} = -\sin\theta_{n-r}\ldots\sin\theta_{r-2+2s}(\sin\theta_{r-3+2s}\cos\theta_{r-4+2s}-i\cos\theta_{r-3+2s}),\\
&\bar{a} = \sqrt{\frac{1+a_0}{1-a_0}}\cos\theta_{n-r}+k,\quad \bar{d}=-\sqrt{\frac{1-a_0}{1+a_0}}\cos\theta_{n-r}+k,\\
&\qquad\qquad s=1,2,\ldots,\frac{n+2}{2}-r.
\end{aligned}\right\}\tag{95}$$

The two sets of flats may then be written:

$$\mathrm{I(c)}\ \left\{\begin{aligned}
x_i &= ay_i+b_i,\\
x_{r+s-2} &= ay_{r+s-2},\\
z &= -\sum_1^{r-2} b_iy_i+d,
\end{aligned}\right.$$

$$\mathrm{II(c)}\ \left\{\begin{aligned}
x_i &= \bar{a}y_i+\bar{b}_i\\
x_{r+s-2} &= \bar{a}y_{r+s-2}+\bar{b}_{r+s-2},\\
z &= \sum_1^{r-2}\bar{b}_iy_i+\Sigma\bar{c}_{r+s-2}y_{r+s-2}+\bar{d},
\end{aligned}\right.\qquad \left\{\begin{aligned}&i=1,2,\ldots,r-2,\\ &s=1,2,\ldots,\frac{n+2}{2}-r\end{aligned}\right\}.$$

Eliminating the parameters θ_i from I(c) we obtain a surface of $\frac{n-6}{2}+r$ dimensions whose equations are

$$\left.\begin{aligned}
&(1+a_0)\sum_1^{r-2}(x_iy_k-x_ky_i)^2+(1-a_0)\sum_1^{r-2}(x_i-ky_i)^2-(1+a_0)\sum_1^{r-2}y_i^2\\
&\qquad\qquad +(1+a_0)(z-k)^2-(1-a_0)=0,\\
&(1+a_0)\sum_1^{r-2}(x_{r+s-2}y_i-x_iy_{r+s-2})y_i-(1+a_0)(z-k)y_{r+s-2}\\
&\qquad\qquad +(1-a_0)(x_{r+s-2}-ky_{r+s-2})=0,\qquad s=1,2,\ldots,\frac{n+2}{2}-r.
\end{aligned}\right\}\tag{96a}$$

Its conjugate, which may be obtained by changing the sign of a_0 and k in the above equations, has for equations:

$$\left.\begin{aligned}
&(1-a_0)\sum_1^{r-2}(x_iy_k-x_ky_i)^2+(1+a_0)\sum_1^{r-2}(x_i+ky_i)^2-(1-a_0)\sum_1^{r-2}y_i^2\\
&\qquad\qquad +(1-a_0)(z+k)^2-(1+a_0)=0,\\
&(1-a_0)\sum_1^{r-2}(x_{r+s}y_i-x_iy_{r+s-2})y_i-(1-a_0)(z+k)y_{r+s-2}\\
&\qquad\qquad +(1+a_0)(x_{r+s-2}+ky_{r+s-2})=0.
\end{aligned}\right\}\tag{96b}$$

The second set of flats which do not lie on the surface intersect those of the first in a point on the surface; the parametric equations of the surface will therefore be found by solving I(c) and II(c) for x_i, y_i, and z. The null-flats of I(c), which also lie on the surface, form part of the intersection of the surface (96a, b) with its conjugate. The second set of null-flats are common tangent flats of the two conjugate surfaces along the remaining locus of intersection in the finite part of space. At infinity there is a common cone of intersection whose equations are:

$$\left.\begin{aligned} &\sum_1^{r-2}(x_i y_k - x_k y_i)^2 = 0, \quad \sum_1^{r-2}(x_{r+1}y_i - x_i y_{r+1})y_i = 0, \\ &x_{r+s-2}y_{r+s'-2} - x_{r+s'-2}y_{r+s-2} = 0, \quad s' \neq s, \quad u = 0. \end{aligned}\right\} \tag{97}$$

The surface (96a) *is the locus of* ∞^{r-2} *flats which intersect a family of* ∞^{n-r} *flats.* Such a locus we shall call a *flat-regulus* and denote it by the symbol $R^{(r-2)}_{\frac{n-6}{2}+r}$, the lower subscript indicating the dimensions of the surface. If a regulus is a quartic $n-2$-spread, or is the intersection of two or more such spreads, we shall call it a *quartic regulus;* it may of course happen, as in the case of (96a) and (96b), that one or more, but not all, reduce to cubic spreads, quadrics or even flats. A cubic (quadric) regulus is a cubic (quadric) $n-2$-spread, or the intersection of two or more such spreads. (96a), (96b) and (90′) are quartic reguli, and (90) a quadric regulus. In 5-space, however, $n=6$, the reguli (96a) and (96b) are cubic, as is also the case in any odd space if $r=3$; the regulus has then only a single infinity of flats.

The flats generating the surfaces belong to certain systems of linear flat-complexes which we shall study in a more comprehensive manner in another paper. We shall state the results obtained in the following theorem which is a generalization of the corresponding one for 3-space given by Lie. It is pertinent to point out here the great generality of the theorem, since in 3-space only one type can exist, namely, $r=1$ for cyclides of the third order, and $r=3$ for those of the fourth:

THEOREM. *The generalized flat-sphere transformation* (3) *carries the Dupin cyclide of the third order and type* $r \lesseqgtr \frac{n-2}{2}$ *in* S_{n-1} ($n-1$ odd) *into a quadric and quartic regulus of* $\frac{n-2}{2}+r$ *dimensions which are conjugate to each other. If* $r=1$ *the quartic regulus is cubic, and in 5-space it reduces to a quadric regulus. If* $r=\frac{n-2}{2}$ *the transform is a quartic* $n-2$*-spread. The*

same transformation carries the Dupin cyclide of the fourth order and type $r \overline{\gtrless} \frac{n+2}{2}$ *into two conjugate quartic reguli of dimensions* $\frac{n+2}{2}+r$, *which become cubic if* $r=3$. *In 5-space they are quadric reguli. If* $r=\frac{n+2}{2}$ *the transform is a quartic* $n-2$*-spread which in 5-space has a quadric locus of double-points at infinity.*

The asymptotic directions proceeding from a point on a regulus $R^{(r-2)}_{\frac{n-6}{2}+r}$ lie in the two flats that intersect at the point. There are $\infty^{\frac{n-4}{2}}$ directions which lie in the flat of the first set, and ∞^{r-3} lying in that of the second set. The asymptotic lines are therefore indeterminate to a less extent than the lines of curvature on a cyclide.* For $n=4$ $r=3$ (ordinary space) the number of directions are finite and equal to 2. The group of $\infty^{\frac{n+1\cdot n+2}{2}}$ contact-transformations whose characteristic functions are given by (2), carries flats into flats, and it therefore transforms all flat-reguli *inter se*. The group is therefore related to all the spreads $R^{(r)}_{\frac{n-2}{2}+r}$ and $R^{(r-2)}_{\frac{n-6}{2}+r}$ in the same way that the projective group of ∞^{15} transformations are related to all the quadric surfaces in 3-space. We have thus obtained a class of surfaces in odd space which from the standpoint of Sphere-Flat Geometry is the generalization of the quadric surface in 3-space, and which we shall meet with again in the study of flat-complexes.

§ 4. *The Asymptotic Lines on a Sphere in Odd Space.*

We have seen that the asymptotic lines on the transform of cyclides in S_{n-1} are to a certain extent indefinite; these spreads are therefore exceptions to the general rule. Thus, for example, the transforms of homofocal or co-axial quadrics have definite asymptotic lines.

The asymptotic lines on a quadric surface have not been determined except for 3-space. Let the quadric be written

$$\Sigma a_i x_i^2 + \Sigma b_i y_i^2 + cz^2 = 1. \tag{98}$$

Transforming by Euler's transformation (79) we have the surface

$$F^2 = \frac{1}{c}\left[1 + c\Sigma \frac{\alpha_i^2}{a_i}\right][1 - \Sigma b_i \beta_i^2], \tag{98'}$$

* This is not true relatively. The regulus is a surface of dimensions $\frac{n-6}{2}+r$ while the corresponding cyclide is of $n-2$ dimensions.

so that the tangential equation of the transform in the α_i, β_i coordinates (5) is known, and the parametric equations of the surface may be derived from it. The surface is not real, and the determination of the lines of curvature is not a simple matter. In the case of a sphere a direct method is successful. Let the sphere be written

$$\Sigma x_i^2+\Sigma y_i^2+z^2=r^2; \tag{99}$$

the system of total differential equations which determine the lines, viz.:

$$dx_i dp_{\frac{n-2}{2}}+dy_i dq_{\frac{n-2}{2}}=0, \quad dq_i dp_{\frac{n-2}{2}}-dp_i dq_{\frac{n-2}{2}}=0, \qquad i=1,2,\ldots,\frac{n-2}{2}, \tag{100}$$

may be replaced by the following:

$$dx_i dp_i+dy_i dq_i=0, \quad dq_1 dp_i-dp_1 dq_i=0, \qquad i=1,2,\ldots,\frac{n-2}{2}. \tag{100'}$$

We have from (99),

$$p_i=-\frac{x_i}{z}, \quad q_i=-\frac{y_i}{z}.$$

Introducing the values of dp_i and dq_i in (100′), and keeping account of the relations $dx_1 dy_i-dy_1 dx_i=0$, $\left(i=2,3,\ldots,\frac{n-2}{2}\right)$, the system becomes

$$\left.\begin{aligned} &dz(x_i dx_i+y_i dy_i)=z(dx_i^2+dy_i^2),\\ &dz^2(y_1x_i-x_1y_i)\;=dz(y_1dx_i-x_1dy_i+x_idy_1-y_idx_1). \end{aligned}\right\} \tag{101}$$

We shall first suppose that $dz \neq 0$. The second set may then be integrated at once giving $\frac{n-4}{2}$ integrals:

$$y_1x_i-x_1y_i=2c_{i-1}z, \qquad i=2,3,\ldots,\frac{n-2}{2}. \tag{102}$$

In order to integrate the first set we shall introduce the variables u_i and v_i, putting

$$\left.\begin{aligned} x_i+iy_i=u_i, \quad x_i=\frac{u_i+v_i}{2},\\ x_i-iy_i=v_i, \quad y_i=\frac{u_i-v_i}{2_i}, \end{aligned}\right\} \tag{103}$$

so that (90) and (102) become

$$z^2+\Sigma u_i v_i=r^2, \quad u_1v_i-v_1u_i=2c_{i-1}z, \qquad i=2,3,\ldots,\frac{n-2}{2}, \tag{104}$$

and the first set (101) is now

$$\frac{dz}{z}=\frac{2du_i dv_i}{v_i du_i+u_i dv_i},$$

which, since $dv_1du_i-du_1dv_i=0$, may be written in the form:

$$\frac{du_i}{dz}=\frac{v_idu_1+u_idv_1}{2zdv_1}.$$

Introducing now a factor of proportionality ρ, we may write this system in the final form:

$$\rho du_i=u_idv_1+v_idu_1,\quad \rho dz=2zdv_1. \tag{105}$$

Differentiating (104) we have,

$$u_1dv_i+v_idu_1-v_1du_i-u_idv_1=2c_{i-1}dz,\quad \Sigma(u_idv_i+v_idu_i)=-2zdz, \tag{106}$$

which, together with (105), constitute a system of n linear and homogeneous equations in $n-1$ unknowns du_i, dv_i, dz. This system will be consistent if, and only if, the determinant of the system vanishes, *i. e.*, if

$$\begin{vmatrix} v_1-\rho & 0 & 0 & \cdots & 0 & u_1 & 0 & 0 & \cdots & 0 \\ v_2 & -\rho & 0 & \cdots & 0 & u_2 & 0 & 0 & \cdots & 0 \\ v_3 & 0 & -\rho & \cdots & 0 & u_3 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & -\rho & 2z & 0 & 0 & \cdots & 0 \\ v_2 & -v_1 & 0 & \cdots & -2c_1 & -u_2 & u_1 & 0 & \cdots & 0 \\ v_3 & 0 & -v_1 & \cdots & -2c_2 & -u_3 & 0 & u_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ v_1 & v_2 & v_3 & \cdots & 2z & u_1 & u_2 & u_3 & \cdots & u_{\frac{n-2}{2}} \end{vmatrix}=0. \tag{107}$$

Expanding this determinant we find that it is a quadratic in ρ, $\rho^{\frac{n}{2}-2}$ being a factor. The zero value of the root may be neglected as no solution of the system (105) corresponds to it. The quadratic equation is:

$$\rho^2+\frac{4z(u_1z+\sum\limits_2^{\frac{n-2}{2}}c_{i-1}u_i)}{\Sigma u_i^2}\rho+\frac{4z^2[\sum\limits_1^{\frac{n-2}{2}}c_{i-1}^2-u_1v_1]}{\Sigma u_i^2}=0. \tag{108}$$

We now solve the equations (104) for v_1 after eliminating $v_2, v_3, \ldots, v_{\frac{n-2}{2}}$ and substitute in (108). Solving we have the two roots

$$\rho=\frac{-2z(u_1z+\sum\limits_2^{\frac{n-2}{2}}c_{i-1}u_i)\pm 2z\sqrt{R}}{\Sigma u_i^2},$$

where $R=\sqrt{u_1^2r^2-\Sigma c_{i-1}^2\Sigma u_i^2+(\Sigma c_{i-1}u_i)^2}$. Introducing ρ in the system (105),

and eliminating the v's by means of (108), we have the following system of differential equations:

$$\frac{dz}{u_1(r^2+z^2)\mp 2z\sqrt{R}}=\frac{du_1}{u_1\Bigl(u_1z+\sum\limits_{2}^{\frac{n-2}{2}}c_{i-1}u_i\mp\sqrt{R}\Bigr)}$$

$$=\frac{du_2}{u_2\Bigl(u_1z+\sum\limits_{2}^{\frac{n-2}{2}}c_{i-1}u_i\mp\sqrt{R}\Bigr)-c_1\Sigma u_i^2}=\dots$$

$$=\frac{du_{\frac{n-2}{2}}}{u_{\frac{n-2}{2}}\Bigl(u_1z+\sum\limits_{2}^{\frac{n-2}{2}}c_{i-1}u_i\mp\sqrt{R}\Bigr)-c_{\frac{n-4}{2}}\Sigma u_i^2}. \tag{109}$$

By properly combining we easily find the following algebraic integrals:

$$\frac{u_{i+2}}{c_{i+1}}=\frac{u_2}{c_1}+k_iu_1, \qquad i=1,2,\dots,\frac{n-6}{2}, \tag{110}$$

the k's being constants of integration. The remaining integrals may now be found as follows: We substitute the values of the u's from (110) in the expression for R, above, and find

$$R=\sqrt{r^2-\Bigl(1+\sum_{1}^{\frac{n-6}{2}}k_i^2\Bigr)\sum_{1}^{\frac{n-4}{2}}c_i^2+\Bigl(\sum_{1}^{\frac{n-6}{2}}k_ic_{i+1}\Bigr)^2}\,u_1=Au_1;$$

we also have

$$\sum_{2}^{\frac{n-2}{2}}c_{i-1}u_i=\sum_{1}^{\frac{n-6}{2}}k_ic_{i+1}u_1+\frac{\Sigma c_i^2}{c_1}u_2=Bu_1+Cu_2,$$

$$\sum_{1}^{\frac{n-4}{2}}u_i^2=(1+\Sigma k_i^2)u_1^2+\frac{2\Sigma k_ic_{i+1}}{c_1}u_1u_2+\frac{\Sigma c_i^2}{c_1^2}u_2^2$$

$$=Du_1^2+\frac{2B}{c_1}u_1u_2+\frac{C}{c_1}u_2^2, \quad r^2-A^2=c_1CD-B^2.$$

We have then to integrate the equations

$$\frac{dz}{u_1(r^2+z^2)\mp 2zAu_1}=\frac{du_1}{u_1(u_1z+Bu_1+Cu_2\mp Au_1)}$$

$$=\frac{du_2}{u_2(u_1z+Bu_1+Cu_2\mp Au_1)-c_1\Bigl(Du_1^2+\dfrac{2B}{c_1}u_1u_2+\dfrac{C}{c_1}u_2^2\Bigr)},$$

which may be further simplified by putting $u_2 = vu_1$:

$$\frac{dz}{r^2+z^2\mp 2Az} = \frac{du_1}{u_1(z+B+Cv\mp A)} = -\frac{dv}{c_1D+2Bv+Cv^2}. \qquad (111)$$

These equations may be integrated by quadratures. We find,

$$z = \frac{\alpha(r^2-A^2)-(Cv+B)}{1+\alpha(Cv+B)} \pm A, \quad u_1 = \frac{\beta}{1+\alpha(Cv+B)}\, u_2 = \frac{\beta v}{1+\alpha(Cv+B)}. \qquad (112)$$

The remaining u's and all the v's are now obtained from the equations (110) and (104). A rather long but not difficult calculation shows that they may be expressed in the form:

$$u_{i+2} = \frac{p_{i+2}+q_{i+2}v}{1+\alpha(Cv+B)}, \quad v_i = \frac{r_i+s_iv}{1+\alpha(Cv+B)}, \qquad i=1, 2, \ldots, \frac{n-2}{2}, \qquad (113)$$

where the coefficients p_i, q_i, r_i, s_i depend on the $n-3$ integration constants c_i, k_i, α, β. *From these equations it appears at once that the integral curves are straight lines.* Through any point on the sphere will pass two such lines corresponding to the two roots of (108), or, corresponding to the $+$ and $-$ sign of A in (111).

We shall investigate the remaining asymptotic loci. We assumed $dz \neq 0$. Let $dz=0$. The second set of equations (101) vanish identically, and the equations of the first set are:

$$dz=0, \quad dx_i^2+dy_i^2=0, \qquad i=1, 2, \ldots, \frac{n-2}{2},$$

which are satisfied if we put

$$\text{I.} \quad z=C, \quad x_i+iy_i=m_i, \quad \text{or}, \quad \text{II.} \quad z=C, \quad x_i-iy_i=n_i.$$

The first set of flats intersect the sphere in an $\frac{n-4}{2}$ dimensional flat whose equations are:

$$z=C, \quad x_i+iy_i=m_i, \quad C^2+\Sigma m_i(x_i-iy_i)=r^2,$$

and the second set intersects the sphere in a similar flat whose equations are:

$$z=C, \quad x_i-iy_i=n_i, \quad C^2+\Sigma n_i(x_i+iy_i)=r^2.$$

Through any point on the sphere pass two such flats and hence there will be $\infty^{\frac{n-4}{2}}$ asymptotic directions through it and lying in the respective flats. We shall call these loci *asymptotic flats.* The results obtained will now be stated in the following

THEOREM. *There are two sets of asymptotic loci on a sphere in odd-dimensional space. The first set consists of a double family of ∞^{n-3} straight*

lines of which through every point on the sphere will pass two. The second set consists of a double family of $\infty^{\frac{n}{2}}\,\frac{n-4}{2}$*-dimensional flats, the asymptotic flats, and through every point on the sphere will pass two flats and* $\infty^{\frac{n-4}{2}}$ *asymptotic directions lying in each flat.*

For 5-space these flats are straight lines so that we may say:

Through every point on a sphere in 5-space pass four definite asymptotic directions and the four sets of asymptotic lines consist of a four-fold family of ∞^3 *straight lines.*

In 3-space the first set of ∞^1 asymptotic lines remain while the asymptotic flats, being of zero dimensions, become the ∞^2 points of the sphere.

There exist an indefinite number of surfaces whose asymptotic loci are straight lines and flats, namely, all the transforms of the sphere by the transformations of the group (2). Thus, the quadrics

$$\Sigma a_i(x_i^2+y_i^2)+cz^2=r^2$$

belong to this class, since the transformation,

$$x_i=a_ix'_i, \quad y_i=a_iy'_i, \quad z=cz',$$

carries flats into flats.

The transform of the sphere in $\bar{\bar{S}}_{n-1}$ is a sextic surface in S_{n-1} whose equation is:

$$[r^2(1+X_{n-1}^2)-\Sigma(X_{2i-1}+iX_{2i})^2][1+X_{n-1}^2+\Sigma(X_{2i-1}-iX_{2i})^2]=X_{n-1}^2(1+\sum_1^{n-1}X_i^2)^2,$$

which has the absolute as locus of double points. The surface has four focal sheets if $n \overline{>} 6$. *In 5-space the surface has circular lines of curvature in all four systems;* it therefore presents a striking resemblance to the cyclides in ordinary space.

WEST VIRGINIA UNIVERSITY, *January* 10, 1916.

Irrational Involutions on Algebraic Curves.

By Joseph Vital DePorte.

§ 1. *General Definitions.*

Given a correspondence between the points of two algebraic plane curves, $F(x_1, x_2, x_3) \equiv F(x) = 0$, of genus p, and $f(y) = 0$, of genus π, such that to an arbitrary point P of the first curve correspond b points of the second, and to an arbitrary point P' of the second correspond a points of the first. The group of b points on $f(y) = 0$, corresponding to P in its turn fixes on $F(x) = 0$ b groups of a points each, and P belongs to each of these groups. Similarly, P' on $f(y) = 0$ belongs to a groups, each consisting of b points. The numbers a, b indicating the number of groups in the correspondence to which arbitrary points on $f(y) = 0$ and $F(x) = 0$ belong, are called the *indices* of the (a, b) correspondence, so that the correspondence on $F(x) = 0$ is of index b, on $f(y) = 0$ of index a.

Every (a, b) correspondence can be expressed by means of two equations between the coordinates of corresponding points, if the two curves are irrational. There is usually a finite number of pairs of points not belonging to the correspondence, the coordinates of which satisfy the equations. If the curves are rational the correspondence can always be expressed by means of one equation. If a point x on $F(x) = 0$ is defined in terms of a parameter λ, and a point y on $f(y) = 0$ in terms of μ, then the (a, b) correspondence may be defined by the polynomial $\Sigma \phi_i(\lambda) \cdot \mu^{b-i}$, in which each $\phi_i(\lambda)$ is a polynomial of order a in λ. It will be stated later when this can be done even when the curves are irrational.

If either a or b is one, the coordinates of P or P' can be expressed rationally in terms of the coordinates of P' or P. If both a and b are equal to one, that is if the correspondence is one-to-one, then it is birational, and the coordinates of points on each curve can be expressed rationally in terms of the coordinates of points on the other. In this case the curves $F(x) = 0$, $f(y) = 0$ are said to be birationally equivalent.

A correspondence of index 1, for instance $(a, 1)$, is called an *involution* of order a. The genus of the curve, the points of which are in $(1, a)$ corre-

spondence with the points of the given curve, is called the *genus of the involution.* If the genus of the involution is zero it is said to be rational; otherwise, irrational.

Every rational involution is a linear series and vice versa.

If a curve possesses two rational or irrational involutions, γ'_m, μ'_n $(m \leq n)$ such that every point of a group of the first belongs to a distinct group of the second, in which case the groups of μ'_n, taken m at a time are conjugate under γ'_m, we say that γ'_m is compounded with μ'_n.

Given a curve possessing a γ'_i. If an arbitrary point determines not 1, but k distinct groups of the involution, the involution is said to be multiple.

§ 2. *Branch Points and Coincidences.*

If in a general (a, b) correspondence two or more points of a group of points corresponding to a given point P of a curve coincide, the point P is called a branch-point. The numbers of simple branch-points y and y' (that is those for which only two of the corresponding points coincide) in an (a, b) correspondence between curves of genera p and π are connected by the following relation, due to Zeuthen:*

$$y-y'=2a(\pi-1)-2b(p-1). \tag{1}$$

If the correspondence is involutorial, for example $(a, 1)$, then $y=0$, and we have the theorem:

The number of double points of an involution of order a, genus π, on a curve of genus p is

$$y'=2(p-1)-2a(\pi-1). \tag{2}$$

If the correspondence between the curves is $(1, 1)$, then (1) furnishes a direct proof of Riemann's theorem concerning the equality of the genera of curves that are birationally equivalent. For, if $a=b=1$, then $y=y'=0$, and we have $2(\pi-1)=2(p-1)$, hence $p=\pi$.

From Zeuthen's formula also follows† that there do not exist involutions of genus $p\,(p>1)$ on curves of the same genus.

For, setting $\pi=p$ in (2),

$$y'=2(p-1)-2a(p-1).$$

But $y'\geq 0$, $\therefore\ p-1\geq a(p-1)$, $\therefore\ a=1$.

---

* H. G. Zeuthen, "Nouvelle démonstration de théorèmes sur les séries de points correspondants sur deux courbes." *Math. Annalen*, Vol. III (1870), pp. 150–156.

† Weber, *Jour. für Math.*, Vol. LXXVI (1876), p. 345.

§ 3. *Valence of Involutions.*

If the curves in (a, b) correspondence coincide, then we have an (a, b) correspondence between points of one curve. If as a point P (P') moves along the curve the b (a) corresponding points, together with the point P (P') counted γ times, $(\gamma \gtreqless 0)$, moves in a linear series, the correspondence is said to be of valence γ. In other words, given P and a group of b points corresponding to it. Let P go into P_1, b into b_1, then if the correspondence is of valence γ,

$$\gamma P + b \equiv \gamma P_1 + b_1 \quad (\gamma \gtreqless 0),$$

or the groups $\gamma P + b$, $\gamma P_1 + b_1$ belong to the same linear series of order $\gamma + b$. If γ is negative, (3) may not have a geometric meaning, but by transposing the term containing γ we get

$$\gamma P_1 + b \equiv \gamma P + b_1 \quad (\gamma > 0),$$

which can be interpreted to mean that $\gamma P_1 + b$, $\gamma P + b_1$ belong to the same linear series of order $\gamma + b$.

The number of coincidences z in a valence correspondence as given by the Cayley-Brill-Hurwitz formula is

$$z = a + b + 2p\gamma. \tag{3}$$

§ 4. *Notation.*

We shall represent involutions by small letters, with subscripts indicating the order and genus, and superscript 1 indicating the dimension, using letters of the Latin alphabet for rational involutions (linear series), and of the Greek alphabet for irrational involutions. Thus, $\gamma'_{a,\pi}$ reads: "An irrational involution of order a genus π." g'_a—"a rational involution, or linear series, of order a." Thus $g'_a \equiv \gamma'_{a,0}$. Capital letters are used to indicate individual groups: Γ_a, G_a.

§ 5. *Application to Cubic Curves.*

Every cubic has a single infinity of rational involutions of order 2. In fact, consider a pencil of lines with vertex at an arbitrary point S of the cubic. Each line of the pencil cuts the cubic in two points, each of which uniquely determines the other, since it fixes a line of the pencil. We have then an involution of pairs of points, and since the vertex S is arbitrary, we can construct a single infinity of such central involutions. The involutions are rational, for if we take the point S as the point $(0, 0, 1)$, for example, the equation of any line of the pencil is of the form $x + \lambda y = 0$. The coordinates of a point P on the cubic fix the value of the parameter λ, and we can express

the coordinates of the point P', conjugate to P in the involution rationally in terms of the parameter.

In general, if the lines joining pairs of points of a simple involution of order 2 pass through a point or envelope a rational curve the involution is rational. We saw the truth of the first statement. In the second case the points of the curve enveloped by the lines are in (1, 1) correspondence with the groups of the involution, since to every point of the curve corresponds a line determining a group of the involution. The genus of the curve is, by definition, the genus of the involution. If the curve is rational, so is the involution.

We can construct on a cubic of genus 1 involutions of order 2 which are not rational by taking the product of two central involutions. Let us take a point $S(s)$, (we shall thus indicate the parameter, in terms of elliptic functions of which the coordinates of a point on the cubic can be expressed rationally) as center and project from it an arbitrary point $P(p)$ on the curve into $P_1(p_1)$. Project then $P_1(p_1)$ from another center $S_1(s_1)$ into $P_2(p_2)$. Repeat the process by projecting P from S_1 into $P'(p')$, and P' from S into $P''(p'')$. If $P_2=P''$, we have an involution of order 2 of which P and P_2 are a pair.

The necessary and sufficient condition that $P_2\equiv P''$ is that the parameters of S and S_1 differ by half a period.

The sum of the parameters of three collinear points on an elliptic curve is congruent to zero.

The points S. P, P_1 are collinear, hence

$$s+p+p_1\equiv 0, \text{ or } p_1\equiv -(s+p).$$

The points P_1, S_1, P_2 are collinear, hence

$$-(s+p)+s_1+p_2\equiv 0, \text{ or } p_2\equiv s+p-s_1.$$

Also the points P, S_1, P' lie on a straight line, hence

$$s_1+p+p'\equiv 0, \text{ or } p'\equiv -(s_1+p).$$

And the points P', S, P'' are collinear, hence

$$-(s_1+p)+s+p''\equiv 0, \text{ or } p''=s_1+p-s.$$

If $P''\equiv P_2$, $p''\equiv p_2$, or $s+p-s_1\equiv s_1+p-s$,

$$\therefore 2s\equiv 2s_1 \ (\text{mod } \omega, \omega'),$$

hence the parameter of S_1 differs from that of S by half a period. Conversely, if s and s_1 differ by half a period, $P''=P_2$. Since the parameter of P does not enter in the last equation the statement is true for any point on the cubic.

If we draw the tangent to the cubic at S it will cut the curve again in one point $O(o)$. From O we can draw three tangents to the cubic, different from the tangent at O and OS. Let the points where the three tangents touch the curve be $S_1(s_1)$, $S_2(s_2)$ and $S_3(s_3)$. We have then the following four relations between the parameters of the points O, S, S_1, S_2, S_3,

$$2s+o\equiv 0,\quad 2s_1+o\equiv 0,\quad 2s_2+o\equiv 0,\quad 2s_3+o\equiv 0.$$

Eliminating o we find that the parameters of S_1, S_2, S_3 differ from the parameter of S by half a period.

It can easily be seen that the converse is also true, namely, if the parameters of two points differ by half a period, the tangents to the cubic at these points intersect in a point on the curve.

Collecting the above results we can state that the necessary and sufficient condition that the product of two central involutions on a cubic of genus 1 is an involution of order 2 is that the tangents to the cubic at the centers of the involutions intersect in a point on the cubic.

Associated with any point $S(s)$ on the cubic there are three points $S_1\left(s+\frac{\omega}{2}\right)$, $S_2\left(s+\frac{\omega'}{2}\right)$, $S\left(s+\frac{\omega+\omega'}{2}\right)$ such that

$$SS_1\equiv S_1S,\quad SS_2\equiv S_2S,\quad SS_3\equiv S_3S.$$

It is important to notice that we get the same three involutions, no matter where S is taken. In fact, let $SS_1\equiv S_1S$, also $C(t)C_1(t_1)\equiv C_1(t_1)C(t)$. Under S a point $P(p)$ goes into $P_1(p_1)$, and P_1 under S_1 goes into $P_2(p_2)$, so that

$$s+p+p_1\equiv 0,\quad s_1+p_1+p_2\equiv 0.$$

Eliminating p_1, we have

$$p+(s-s_1)+p_2\equiv 0.$$

If the point P goes under C into $P'(p')$ and P' under C_1 goes into $P''(p'')$, we have:

$$t+p+p'\equiv 0,\quad t_1+p'+p''\equiv 0.$$

Hence, eliminating p',

$$p+(t-t_1)+p''\equiv 0.$$

Since SS_1, and CC_1 are, by hypothesis, involutions,

$$s-s_1\equiv t-t_1.$$

Therefore $p_2\equiv p''$, and the point P has the same conjugate in both involutions. The involutions are, therefore, identical.

The lines joining pairs of corresponding points envelope a curve of genus 1—the genus of elliptic functions in terms of which the equations of the lines can be expressed rationally. Hence, if the product of two central involutions on a cubic curve of genus 1 is an involution of order 2 its genus is 1.

§ 6. *General Theorems.*

Rational involutions have been studied in detail.* Comparatively little has been done in the field of irrational involutions. Castelnuovo † derived the following theorem: *A group of a* $\gamma'_{a,\pi}$ *on a curve of genus* p *has but* $a-1$ *conditions to belong to a group of a* g^r_n, *if* $n-r<p-a\pi$. For $\pi=0$ and $g^r_n \equiv g^{p-1}_{2p-2}$ the theorem reduces to the Riemann-Roch theorem.

Amodeo ‡ derived from the Zeuthen and Cayley-Brill-Hurwitz formulas a number of theorems on the range of possible involutions on curves of given genus. In so far as the theorems refer to irrational involutions on curves of genus greater than 1, and of general moduli they are of no value, since, as will be pointed out later, such involutions do not exist.

In a later paper by Castelnuovo § appears the important theorem:

The necessary and sufficient condition in order that a simply infinite series γ'_a *of order* a, *and of index* b, *on a curve* $F(x)=0$ *of order* n, *genus* p, *belongs to a linear series* g'_a *of the same order, is that the series shall possess* $2b\,(a+p-1)$ *double points.*

The proof, in brief, is as follows:

Construct on $F(x)=0$ a non-special linear series (that is a series g^r_m, where $m-r=p$. It can be cut out by adjoints of order greater than $n-3$), of dimension $a-1$. The difference between the order and dimension of a non-special series being p, its order will be $a-1+p$. The series can always be selected in such a way that no given complete group of a points of γ'_a belongs to a group of the new series. For, we can choose $a-1$ points of a group of γ'_a; and add to them p points taken arbitrarily, but so as not to contain the a-th point of the group. We will have then a group of the series g^{a-1}_{a-1+p}. But this one group will fix the series. Since every group will contain the same p points and $a-1$ other points, we will have constructed a series of the kind desired.

Applying the Segre formula ‖ we find that the number of groups of a points that the linear series g^{a-1}_{a-1+p} and γ'_a have in common is

$$z=b\,(a+p-1)-1/2d,$$

* For the general theory on linear series see Clebsch-Lindemann, "Vorlesungen über Geometrie" (1876); Severi, "Lezioni di Geometria Algebrica" (1910, lith.). A list of references to the recent literature is given in Doehlemann's "Geometrische Transformationen," zweiter Teil (1908), p. 174.

† G. Castelnuovo, "Alcuni osservazioni sopra le serie irrazionali di gruppi di punti appartenenti ad una curva algebrica," *Rom. Acc. Lincei Rend.*, s. 4, Vol. VII² (1891), pp. 294–299.

‡ F. Amodeo, "Contribuzione alla teoria delle serie irrazionali involutorie giacenti sulle varieta algebriche ad una dimensione," *Ann. di Mat.*, s. 2, Vol. XX (1892), pp. 227–235.

§ G. Castelnuovo, "Sulle serie algebriche di gruppi di punti appartenenti ad una curva algebrica," *Rom. Acc. Lincei Rend.*, s. 5, Vol. XV (1906), pp. 337–344.

‖ C. Segre, "Sulle varieta algebriche di una serie semplicemente infinita di spazi," *Rom. Acc. Lincei Rend.*, (4), Vol. III² (1887), pp. 149–153.

where d is the number of double points of γ'_a. It follows that

$$d=2b(a+p-1)-2z.$$

Since $z\geq 0$,

$$d\leq 2b(a+p-1).$$

If d has the value given by the equality sign, z will be equal to zero, and the linear series g^{a-1}_{a-1+p} will not contain any groups of γ'_a. But if a series g^{a-1}_{a-1+p} be constructed to contain in one of its groups a group of γ'_a (which is possible in ∞^{p-1} ways), then the series will contain γ'_a entirely.

Now construct another linear series, g^a_{a+p}, a group of which is to be made up of a group of γ'_a, Γ_a, and p arbitrary points: $c_1, \ldots, c_p$. Then the series g^{a-1}_{a-1+p}, residual with respect to c_i will contain Γ_a, hence all other groups of γ'_a. In other words the groups residual to the groups of γ'_a with respect to g^a_{a+p} will all pass through c_i, where $i=1, 2, \ldots, p$. Hence we have a group G_p residual to any Γ with respect to g^a_{a+p}. In consequence, γ'_a belongs to linear series g_a, which is residual to G_p with respect to g^a_{a+p}.

Conversely, if γ'_a belongs to a linear series g_a, then a linear series g^{a-1}_{a-1+p} which contain one group Γ will contain all. If a series g^{a-1}_{a-1+p} be constructed so as not to contain in any of its groups a given group of γ'_a, it will not contain any group of γ'_a. Hence z will be equal to zero, and $d=2b(a+p-1)$.

Stated in other words, the theorem given as the necessary and sufficient condition that an algebraic correspondence (a, b) between two curves of genera p, π can be expressed by means of a single equation (rational in the coordinates of corresponding points) is that the number of branch-points on one curve is $2b(a+p-1)$, and on the other $2a(b+\pi-1)$, and conversely.

If the correspondence is involutorial, for instance $(a, 1)$, then the series γ'_a is a rational series g'_a, if it has $2(a+p-1)$ double points. This can be seen also from Zeuthen's formula, for, if in (2) we set $y'=2(a+p-1)$, we get $\pi=0$; and, conversely, if $\pi=0$, $y'=2(a+p-1)$. Thus, a central involution on a non-singular cubic, $(a=2, p=1)$ has four double points and is rational, while the involution obtained by taking the product of two central involutions has no double points and is of genus 1.

§ 7. *Statement of the Problem.*

The purpose of this paper is two-fold:

I. To find the range of all possible involutions on curves of given characteristics.

II. Given an involution, to determine the restrictions on a curve of given genus that it may possess this involution.

In order to ascertain the genus of an involution of order a with a given number of coincidences on a curve $F(x)=0$ of genus p, it will suffice to find any curve $f(y)=0$ in $(1, a)$ correspondence with $F(x)=0$. If the genus of $f(y)=0$ is π, the involution will be of genus π. The curve $F(x)=0$ can not have another involution of the same order and the same number of coincidences but different genus. For, suppose it has beside $\gamma'_{a,\pi}$ also $\gamma'_{a,\pi'}$, that is, let there be a curve $\phi(y')=0$ of genus π' also in $(1, a)$ correspondence with $F(x)=0$. Then,

$$2(p-1)-2a(\pi-1)=2(p-1)-2a(\pi'-1), \therefore \pi=\pi'.$$

We shall arrange involutions according to the genus of the curve $F(x)=0$. Every curve of genus p, $p=3\pi+(0, 1, 2)$, if not hyperelliptic, can be reduced to a curve of order not greater than $2\pi+2$, $2\pi+3$, $2\pi+4$ with $2\pi(\pi-1)$, $2\pi^2$, $2\pi^2+2\pi+1$ double points, and we shall, consequently, consider for every genus curves of the lowest order, the so-called normal curves. Hyperelliptic curves will be treated separately.

§ 8. *Involutions on Rational Curves.* $p=0$.

Setting $p=0$ in (2) we get:

$$y'=-2-2a(\pi-1)\,;\ \text{since } y'\geq 0,\ -1-a(\pi-1)\geq 0,$$
$$\therefore a(1-\pi)\geq 1. \text{ But } a>0, \therefore \pi=0.$$

Hence: *Irrational involutions do not exist on rational curves.**

An interesting application of this theorem is found by studying the asymptotic lines of certain ruled surfaces. Given a ruled surface of order $m+n$, having one m-fold directrix line and an n-fold directrix line. The genus of a plane section is

$$\frac{(m+n-1)(m+n-2)}{2}-\frac{m(m-1)}{2}-\frac{n(n-1)}{2}-\Psi=(m-1)(n-1)-\Psi,$$

where Ψ is the number of double generators.

The asymptotic lines are all algebraic and each belongs to a linear complex containing the congruence defined by the directrices. Every generator meets each asymptotic line in two points, and a plane section in one point. Hence, by our theorem we have examples of curves belonging to a linear complex that are not rational.†

---

*For a different proof of the same theorem see Lüroth, "Beweis eines Satzes über rationale Curven," *Math. Annalen*, Vol. IX (1876), p. 163.

† See C. P. Steinmetz, "On the Curves Which are Self-Reciprocal in a Linear Null System, and Their Configurations in Space," American Journal of Mathematics, Vol. XIV (1892), pp. 161–186; V. Snyder, "Asymptotic Lines on Ruled Surfaces Having Two Rectilinear Directrices," *Bulletin American Mathematical Society*, Vol. V (1899), pp. 343–353, and "Twisted Curves Whose Tangents Belong to a Linear Complex," American Journal of Mathematics, Vol. XXIX (1907), pp. 279–288. Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces" (1906), pp. 204–220.

§ 9. *Hyperelliptic Curves.*

Curves of genera 1 and 2 belong to the class of hyperelliptic curves, and it will be appropriate to take up at this point the study of involutions on hyperelliptic curves generally. This was done by Torelli,* and we shall reproduce his main results.

Irrational involutions on hyperelliptic curves are hyperelliptic. That is, if a curve $f(y)=0$ is in $(1, a)$ correspondence with a hyperelliptic curve $F(x)=0$ it is itself hyperelliptic. For, as is known, the $\gamma'_{a,\pi}$ on $F(x)=0$ is compounded with the g'_2 of the curve. The pairs of groups of $\gamma'_{a,\pi}$ conjugate under g'_2 form a rational involution on the double line of $F(x)=0$. The groups of the involution are in (1, 2) correspondence with the points of $f(y)=0$. The latter, then, has a g'_2, and is hyperelliptic.

In particular, curves of genera 1 or 2 can not have irrational involutions other than of genus 1. For, setting $p=1$ in (2), we have:

$$y'=-2a\pi+2a,$$

$$\pi=\ \ 1-\frac{y'}{2a},\ \therefore \pi=0 \text{ or } 1.$$

If $p=2$,

$$y'=2-2a\pi+2a,$$

$$\pi=1-\frac{y'-2}{2a},\ \therefore \pi=0 \text{ or } 1.$$

Given a hyperelliptic curve of order $2\pi+2$, genus π,

$$y_1^2=\prod_{i=1}^{2\pi+2}(x_1-\alpha_i),\quad \alpha_i\neq\alpha_k. \tag{4}$$

Applying the transformation

$$x_1=\frac{f(x)}{\phi(x)},\quad y_1=\frac{y}{[\phi(x)]^{\pi+1}}, \tag{5}$$

where $f(x)$, $\phi(x)$ are relatively prime polynomials of degree a we get the hyperelliptic curve

$$y=\prod_{i=1}^{2\pi+2}[f(x)-\alpha_i\phi(x)]=R(x), \tag{6}$$

where $R(x)$ is a polynomial of order $a(2\pi+2)$. The hyperelliptic curves (4) and (6) are in $(1, a)$ correspondence. The curve (6) has an involution (hyperelliptic) of order a, genus π, a $\gamma^1_{a,\pi}$ which is represented on the x-axis $(y=0)$ on which (6) is mapped doubly by the rational involution,

$$f(x)-x_1\phi(x)=0.$$

---

* R. Torelli, "Sulle involuzioni irrazionali nelle curve iperellitiche," *Palermo Rend.*, Vol. XIX (1905), pp. 297–304.

To determine the genus of the curve (6) we notice that the factors of $R(x)$ have no common roots. All or some of them may have multiple roots, of even or odd multiplicity. In that case

$$R(x)=[S(x)]T(x), \tag{6'}$$

where $T(x)$ is a polynomial of order $2p+1$ or $2p+2$, with simple roots only. Transforming (6) birationally by means of

$$x=x', \quad y=y'S(x'),$$

we obtain

$$y'^2=T(x)=\prod_{i=1}^{\substack{2p+1 \\ \text{or } 2p+2}}(x-b_i), \quad b_i \neq b_k. \tag{7}$$

(7) is of genus p. (6) is in (1, 1) correspondence with it. Hence the genus of (6) is also p.

The coincidences of the curve (6) (which are the double points of g_2') form $2\pi+2$ groups of a rational involution I_a on the x-axis which are made up of the roots of (7) each counted an odd number of times (≥ 1), and the roots of $S(x)$, different from the roots of $T(x)$ (6′), each counted an even number of times (≥ 2). We have at once the following theorem:

The necessary and sufficient condition in order that a hyperelliptic curve $F(x)=0$ of genus p contain an irrational (hyperelliptic) involution of order a, and genus π is that of the $2p+2$ coincidences on the line on which the curve is mapped doubly, each counted an odd number of times and, if necessary, with other points, each counted an even number of times it shall be possible to form $2\pi+2$ groups of a rational involution of order a.

The condition is sufficient. For, let the x-axis on which the hyperelliptic curve $F(x)=0$ is mapped doubly, contain an involution I_a' satisfying the given condition. Then if $f(x)-\alpha_1\phi(x)=0$, $f(x)-\alpha_2\phi(x)=0 \ldots f(x)-\alpha_{2\pi+2}\phi(x)=0$ are the $2\pi+2$ groups, the curve $y^2=\prod_{i=1}^{2\pi+2}[f(x)-\alpha_i\phi(x)]$ has the same group of coincidences as $F(x)=0$, and is rationally equivalent to it. Since the former curve is in $(a, 1)$ correspondence with $y^2=\prod_{i=1}^{2\pi+2}(x-\alpha_i)$, $F(x)=0$ will also be in $(a, 1)$ correspondence with it, and will have a $\gamma'_{a,\pi}$.

Conversely, if $F(x)=0$ has a $\gamma_{a,\pi}$, it can be put in birational correspondence with $y^2=\prod_{i=1}^{2\pi+2}[f(x)-\alpha_i\phi(x)]$ by a proper choice of the constants α_i, and the polynomials $f(x)$ and $\phi(x)$. The coincidences of the two curves on the x-axis will then be projective.*

* Segre, "Introduzione alla geometria sopra un ente algebrico semplicemente infinito," *Ann. di Mat.*, serie 2, Vol. XXII (1894), § 67, note.

But this is the condition stated in the theorem.

The particular case of cyclic irrational involutions on hyperelliptic curves has been studied by Wiman *.

§ 10. *Non-Hyperelliptic Curves.*

Non-hyperelliptic curves of general moduli have only valence-correspondences, † hence they do not have irrational involutions. In order for a curve to possess an involution the moduli of the curve must be specialized. We shall commence the study of involutions on curves with specialized moduli with those of the lower order, namely, 2.

§ 11. *Irrational Involutions of Order 2.*

An irrational involution of order 2 on a given curve associates the points of the curve in pairs, so that to a point P corresponds a unique point P', which is the conjugate of P in the involution. To the point P' corresponds the point P. Thus the involution defines a birational transformation of period 2 of the curve into itself. In other words, if a curve possesses a γ'_2 it must remain invariant under a birational transformation of period 2.

We shall consider first irrational involutions of period 2 on curves which remain invariant under the simplest of birational transformations—linear transformations.

§ 12. $p=3$.

The normal form of a non-hyperelliptic curve of genus 3 is a non-singular quartic. Since under any birational transformation that leaves the curve invariant the system of adjoints of order $n-3$, that is straight lines, goes over into a system of adjoints of the same order, the transformation is linear.

Consider a non-singular quartic invariant under the linear transformation:

$$L \equiv \begin{pmatrix} x_1 & x_2 & x_3 \\ -x_1 & x_2 & x_3 \end{pmatrix} \text{‡}.$$

Its equation will be of the form

$$x_1^4 + x_1^2\phi_2(x_2, x_3) + \phi_4(x_2, x_3) = 0, \tag{8}$$

where the ϕ_i's are homogeneous polynomials in x_2 and x_3 of degree i.

* A. Wiman, "Über die hyperelliptischen Curven und diejenigen vom Geschlecht $p=3$ welche eindeutige Transformationen in sich zulassen," *Bihang till K. Svenska Vet. Akad. Handlingar*, Vol. XXI (1895).

† A. Hurwitz, Über algebraische Correspondenzen und das verallgemeinerte Correspondenzprincip," *Math. Annalen*, Vol. XXVIII (1887), p. 560.

‡ Every linear transformation in the plane of period 2 can be put into this form. A. Hurwitz, 'Ueber diejenigen algebraischen Gebilde, welche eindeutige Transformationen in sich zulassen," *Math. Annalen*, Vol. XXXII (1888), p. 290.

A straight line joining a pair of points P and P' conjugate in the involution will cut the quartic again in two points Q and Q'. Since the points P and P' interchange under the transformation which leaves the quartic invariant, the line PP' will go into itself. Hence, Q, Q' also interchange, in other words, Q and Q' are a pair of the involution. The point P determines not only its conjugate P', but also another pair of the involution. Since P is arbitrary, every point on the quartic determines two pairs of the involution, or

If a curve of genus 3 *has a* γ_2', *the involution is multiple.*

Since we assume that C_4 is not hyperelliptic, it can not have a g_2'. The lines joining pairs of points of the involution belong to a pencil with vertex at the center of homology (1, 0, 0) not on the curve.

To find the genus of the involution we need only construct a curve in (1, 2) correspondence with (8). By means of the transformation

$$T \equiv \begin{pmatrix} x_1 = \sqrt{y_1 y_3}\,, \\ x_2 = y_2\,, \\ x_3 = y_3\,, \end{pmatrix}$$

(8) goes over into the quartic

$$y_1^2 y_3^2 + y_1 y_3 \phi_2(y_2, y_3) + \phi_4(y_2, y_3) = 0, \tag{8'}$$

which is in (1, 2) correspondence with (8). (8') has a tacnode at the point (1, 0, 0) and no other multiple points; it is, therefore, of genus 1.

An involution of order 2 *on a non-hyperelliptic curve of genus* 3 *is of genus* 1.

We can make use of Zeuthen's formula (2) to verify the result. $\gamma_{2,1}'$ on the quartic has four coincidences—the points of intersection of the line $x_1 = 0$ with the curve. If in (2) we put $y' = 4$, $p = 3$, $a = 2$, we have

$$4 = 2(3-1) - 2 \cdot 2(\pi - 1), \quad \text{or} \quad \pi = 1.$$

§ 13. $p = 4$.

Normal form of a non-hyperelliptic curve of genus 4 is a quintic with two double points. The equation of a quintic with two double points at (0, 1, 0) and (0, 0, 1) invariant under L is of the form

$$x_1^4 \phi_1(x_2, x_3) + x_1^2 \phi_3(x_2, x_3) + a x_2^3 x_3^2 + b x_2^2 x_3^3 = 0. \tag{9}$$

The center (1, 0, 0) of the homology is a simple point on the quintic. A line joining a pair of points in the involution passes through (1, 0, 0) and cuts the curve again in two points Q and Q'. By the same method as in the previous case we may therefore state the theorem:

If a curve of genus 4 has a γ_2', *and is invariant under a linear transformation, the involution is multiple.*

Under T (9) goes over into the quartic

$$y_1^2 y_3 \phi_1(y_2, y_3) + y_1 \phi_3(y_2, y_3) + a y_2^3 y_3 + b y_2^2 y_3^2 = 0. \qquad (9')$$

(9′) is of genus 2, since it has one double point at (1, 0, 0).

An involution of order 2 *on a curve of genus* 4 *is of genus* 2.

The line $x_1=0$ cuts the quintic in one point besides the double points (0, 1, 0) and (0, 0, 1). The center (1, 0, 0) is also a coincident point. Hence the involution has two coincidences. Setting in (2) $y'=2$, $p=4$, $a=2$, we have,

$$2=2(4-1)-2\cdot 2(\pi-1), \text{ or } \pi=2.$$

§ 14. $p=5$.

Curves of genus 5, which do not possess a g_3' can be reduced to a sextic with five double points. A curve of genus 5 having a g_3' can be reduced to a quintic with one double point.

a. If a sextic remains invariant under L and does not pass through the center of homology (1, 0, 0), its equation is of the form

$$x_1^6 + x_1^4 \phi_2(x_2, x_3) + \ldots = 0.$$

If the center of homology is on the sextic it is a double point, for the equation of the curve is then of the form

$$x_1^4 \phi_2(x_2, x_3) + \ldots = 0.$$

In either case, since lines passing through the center of homology and joining pairs of points conjugate in the involution, go over into themselves under the transformation which interchanges the points, every point on the sextic determines in the first case three, and in the second case two groups of the involution.

b. The equation of a quintic of genus 5 invariant under L is of the form

$$x_1^4 \phi_1(x_2, x_3) + \ldots = 0,$$

the double point being on the axis of homology $x_1=0$. The center of homology (1, 0, 0) is on the curve. A line joining a pair of points conjugate in the involution cuts the curve in another pair of points, which also belongs to the involution. We may conclude, then, that *if a curve of genus* 5 *has a* γ_2', *and is invariant under a linear transformation, the involution is multiple.*

§ 15. *Involution of Order* 2 *on Curves of Any Genus.*

In general, *if a curve of any genus greater than* 1, *not hyperelliptic, and of any order, has a* γ_2' *and is invariant under a linear transformation, the invo-*

lution is multiple. For, suppose the order of the normal curve to which the given curve can be reduced by birational transformations is n. If n is even, $n=2m$, the normal curve either does not pass through the center of homology $(1, 0, 0)$, or has at the center a singular point of even multiplicity. The equation of the curve is of the form

$$x_1^{2(m-k)}\phi_{2k}(x_2, x_3)+\ldots\ldots=0,$$

where $k\geqq 0$. Every line through the center of homology cuts the curve in $2(m-k)$ points which interchange in pairs under L. An arbitrary point on the curve thus determines $m-k$ groups of the involution.

If n is odd, $n=2m+1$, the equation of the normal curve is of the form

$$x^{2(m-r)}\phi_{2r+1}(x_2, x_3)+\ldots\ldots=0,$$

where $r\geqq 0$. The center of homology is on the curve. Every line through it cuts the curve in $2(m-r)$ points. An arbitrary point on the curve determines $m-r$ groups of the involution.

§ 16. *Equations of Transformation.*

In order to determine the genera of involutions of order 2 on given curves, it is convenient to view the transformation which carries the given curve into a curve in $(1, 2)$ correspondence with it geometrically. If we consider a new plane (y_1, y_2, y_3) such that to any point (x_1, x_2, x_3) in A corresponds one point (y_1, y_2, y_3) in A', but to any point (y_1, y_2, y_3) in A' correspond two points in A, we may write

$$T^{-1}\equiv\begin{pmatrix} x_1^2=y_1 \\ x_2x_3=y_2 \\ x_3^2=y_3 \end{pmatrix}, \quad\text{and}\quad T\equiv\begin{pmatrix} x_1=\sqrt{y_1y_3} \\ x_2=y_2 \\ x_3=y_3 \end{pmatrix}.$$

T sets up a $(2, 1)$ correspondence between the plane A of the given curve and the plane A' of the curve in $(1, 2)$ correspondence with it. The fundamental elements in the A-plane are one fundamental point, $(0, 1, 0)$, and one fundamental line, $x_3=0$; in the A'-plane there are also one fundamental point, $(1, 0, 0)$, and one fundamental line, $y_3=0$. To the system of lines in the A-plane, $x_1+ax_2+bx_3=0$ corresponds in the A'-plane a system of conics $y_1y_3+(ay_2+by_3)^2=0$ passing through the fundamental point $(1, 0, 0)$ and tangent to the fundamental line $y_3=0$. The basis points of the system of conics are the points of intersection of the conics $y_1y_3+ay_2^2$ and $y_3(y_1+by_3)=0$. The conics of the net have three-point contact at $(1, 0, 0)$, hence one free point of intersection which corresponds to a point of intersection of two lines in the A-plane.

To the system of lines in the A'-plane, $y_1+ay_2+by_3=0$, corresponds in the A-plane the system of conics $x_1^2+ax_2x_3+bx_3^2=0$. The conic $x_1^2+ax_2x_3=0$ and the line-pair $x_3^2=0$ have two fixed points of intersection at the fundamental point $(0, 1, 0)$; the system of conics in the A-plane has therefore two free points of intersection, which correspond to a point of intersection of two lines in the A'-plane.

To a curve of order n (not passing through the fundamental point) generated in the A-plane by a point-pair P, P_1, corresponds in the A'-plane a curve of order n counted twice. If the curve C_n in the A-plane passes k times through the fundamental point $(0, 1, 0)$, its image is a curve of order $n-\frac{k}{2}$ counted twice, together with the fundamental line $y_3=0$ counted k times.*

If C_n cuts the fundamental line $x_3=0$ in n distinct points, its image passes $\frac{n}{2}$ times through the fundamental point $(1, 0, 0)$. When two or more points of intersection coincide, that is if C_n has a multiple point on $x_3=0$, a corresponding number of tangents to the image curve at the point $(1, 0, 0)$ coincide.

If C_n does not pass through the center of homology $(1, 0, 0)$ its equation is of the form

$$x_1^{2m}+x_1^{2(m-1)}\phi_2(x_2, x_3)+\ldots\ldots=0,$$

$n=2m$. If n is odd, $n=2m+1$, the center of homology is on C_n. Under T C_n goes over into

$$y_1^m y_3^m+y_1^{m-1}y_3^{m-1}\phi_2(y_2, y_3)+\ldots\ldots=0.$$

Hence, if a curve of order $2m$ in the A-plane does not pass through the point $(1, 0, 0)$, its image in the A'-plane has m branches touching each other at the point $(1, 0, 0)$.

The center of homology lies on C_n if n is odd, in which case C_n passes through it an odd number of times (≥ 1), or, if n is even, when the center of homology is a singular point on the curve of even multiplicity (≥ 2).

If C_n has a singular point of order k on the axis of homology $x_1=0$, we can without loss of generality take the point as the point $(0, 0, 1)$. If k is even, $k=2s$, the equation of C_n is of the form

$$x_3^{n-2s}\phi_s(x_1^2, x_2^2)+\ldots\ldots=0.$$

By means of T we get as the image of C_n the curve

$$y_3^{n-2s}\phi_s(y_1, y_3, y_2^2)+\ldots\ldots=0,$$

* For a comprehensive treatment of birational transformations see K. Doehlemann, "Geometrische Transformationen," Vol. II (1908).

which has s branches touching the line $y_1=0$ at the point $(0, 0, 1)$. If k is odd, $k=2t+1$, the equation of C_n is of the form

$$x_2 x_3^{n-(2t+1)} \phi_t(x_1^2, x_2^2) + \ldots . = 0,$$

and the image of C_n is the curve

$$y_2 y_3^{n-(2t+1)} \phi_t(y_1, y_3, y_2^2) + \ldots . = 0,$$

which has t branches touching each other at the point $(0, 0, 1)$.

The nature of the transformation T requires that the singularities of the curve in the A-plane, barring those in one of the above-discussed exceptional positions, occur in even numbers of similar ones, one pair of which gives rise to one similar singularity on the image of the curve in the A'-plane.

§ 17. *Illustration.*

Let us now determine the genus of a γ_2' on a sextic of genus 5 and not having a g_3', hence having five double points. Since the number of double points is odd, one of them has to be taken in an exceptional position, while the remaining four give rise to two double points on the image curve in the A'-plane. If we take the double point at the fundamental point, the image of the sextic will be a quintic. The sextic cuts the fundamental line in four points besides the fundamental point, hence two branches of the quintic touch each other at the fundamental point in the A'-plane. The latter singularity is equivalent to two double points. The quintic, then, has four double points, and is of genus 2. Algebraically we get the same result by noticing that the equation of a sextic not passing through the center of homology $(1, 0, 0)$ and having a double point at $(0, 1, 0)$ is of the form

$$x_2^4(a x_1^2 + b x_3^2) + \ldots . + x_1^6 = 0.$$

Under T it goes over into the quintic

$$y_2^4(a y_1 + b y_3) + \ldots . + y_1^3 y_3^2 = 0,$$

which has two consecutive double points at $(1, 0, 0)$.

Had we taken the double point on the axis of homology, the result would be the same. In fact consider a sextic with a double point at $(0, 0, 1)$. It cuts the fundamental line in six points. Its image is a sextic having three branches touching each other at the fundamental point in the A'-plane. This singularity is the equivalent of six double points. The sextic in the A'-plane has eight double points, it is of genus 2.

Consider now the case when the normal form of a curve of genus 5 is a quintic with one double point. The center of homology is on the quintic, and if we take the double point as the point $(0, 1, 0)$, the image of the quintic is a quartic with one double point, hence of genus 2.

An involution of order 2 *on a curve of genus* 5 *is of genus* 2.

§ 18. *General Involutions of Order* 2.

We can now proceed to the general case. Given C_n. In order to possess a γ_2' its equation must remain invariant under some birational transformation of period 2. If the curve is non-singular, hence of maximum genus $=\dfrac{(n-1)(n-2)}{2}$, the transformation must be linear.* If n is even, the equation of the curve is of the form

$$x_1^n+x_1^{n-2}\phi_2(x_2, x_3)+\ldots+\phi_n(x_2, x_3)=0.$$

Its image under T is

$$y_1^{\frac{n}{2}}y_3^{\frac{n}{2}}+y_1^{\frac{n-2}{2}}y_3^{\frac{n-2}{2}}\phi_2(y_2, y_3)+\ldots+\phi_n(y_2, y_3)=0,$$

a curve of order n with two consecutive $\dfrac{n}{2}$-fold points at the point (1, 0, 0). The genus of the curve is

$$\frac{(n-1)(n-2)}{2}-\frac{n}{2}\left(\frac{n}{2}-1\right)=\frac{(n-2)^2}{4}.$$

Let us apply as a check Zeuthen's formula. The number of coincidences of γ_2' is n, the points of intersection of $x_1=0$ with the curve,

$$\therefore n=2\left[\frac{(n-1)(n-2)}{2}-1\right]-4(\pi-1)\,; \quad \therefore \pi=\frac{(n-2)^2}{4}.$$

If we denote $\dfrac{n}{2}$, the number of groups of γ_2' on a line through the center of homology by r, we can write the genus of the involution in the form $(r-1)^2$.

The genus of γ_2' on a non-singular curve of even order is $(r-1)^2$, where r is the number of groups of γ_2' on a line passing through one group of the involution.

If the order of the curve is odd, its equation is of the form

$$x_1^{n-1}\phi_1(x_2, x_3)+x_1^{n-3}\phi_3(x_2, x_3)+\ldots+\phi_n(x_2, x_3)=0.$$

Its image in the A'-plane is

$$y_1^{\frac{n-1}{2}}y_3^{\frac{n-1}{2}}\phi_1(y_2, y_3)+\ldots+\phi_n(y_2, y_3)=0,$$

a curve of order n. The genus of the curve is $\dfrac{(n-1)(n-3)}{4}=r(r-1)$, where $r=\dfrac{n-1}{2}$, and is, as above, the number of groups of γ_2' on a line containing one group.

* V. Snyder, "On Birational Transformations of Curves of High Genus," American Journal of Mathematics, Vol. XXX (1909).

The coincidences of γ_2' are the n points of intersection of $x_1=0$ with the curve and the point (1, 0, 0). By Zeuthen's formula,

$$n+1=2\cdot\left[\frac{(n-1)(n-2)}{2}-1\right]-4(\pi-1), \quad \therefore \pi=\frac{(n-1)(n-3)}{4}.$$

The genus of γ_2' on a non-singular curve of odd order is $r(r-1)$, where r is the number of groups of γ_2' on a line passing through one group of the involution.

Thus, on a non-singular quartic γ_2' is of genus $(2-1)^2=1$, on a non-singular quintic of genus $2\cdot(2-1)=2$, on a non-singular sextic of genus $(3-1)^2=4$, and so on.

If the given curve $F(x)=0$ is non-hyperelliptic, and has the equivalent of not more than $E\left(\frac{n-1}{2}\right)^2-3$ double points, $E(k)$ being the largest integer less than k, it remains invariant under linear transformations only.*

If the number of singularities is greater, the curve may remain invariant under transformations of period 2 other than linear. In that case, however, there exists a curve birationally equivalent to $F(x)=0$, the equation of which contains only even powers of one of the variables, say x_1, $f(x_1^2, x_2, x_3)\equiv f^2(x)=0$, and which is therefore transformed into itself by L.†

Consequently, a curve possessing a γ_2' is either itself invariant under L or can be put into (1, 1) correspondence with a curve which is invariant under L. In the first instance, as we have seen, if $F(x)=0$ is not hyperelliptic, the involution is multiple. If the involution defines on $F(x)=0$ a transformation which is not linear, consider $f^2(x)=0$, identical with $F(x)=0$ as regards involutions. The pencil of lines through the center of homology cuts the curve in k pairs of points of the involution. If $k=1$, the involution is a rational g_2', and the curve is, therefore, hyperelliptic. If $k>1$, the involution is multiple. Hence we can generalize the theorem stated in § 15:

A curve having a simple γ_2' is hyperelliptic. If a non-hyperelliptic curve has a γ_2' the involution is multiple.

We can readily determine the genus of a γ_2' on a curve of given characteristics. Consider $F(x)=0$, of genus p. The singularities of the curve, not in exceptional positions, appear in pairs of similar ones. They are equivalent to $2\sum_{\substack{i=2,\dots\\ v=1,\dots}} v_i\cdot\frac{i(i-1)}{2}$ double points, where $2v_i$ denotes the number of i-fold points.

* V. Snyder, "On Birational . . . .," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1909).
† A. Hurwitz, "Ueber diejenigen . . . .," *Math. Annalen*, Vol. XXXII.

They give rise on $f(y)=0$ in the A'-plane to singularities equivalent to $\sum\limits_{\substack{i=2\ldots\\ v=1\ldots}} v_i\cdot\frac{8i(i-1)}{2}$ double points. If $F(x)=0$ has no other singularities, and does not pass through the center of homology $(1, 0, 0)$, $f(y)=0$ will in addition have $\frac{n}{2}$ branches touching each other at $(1, 0, 0)$. Its genus, and therefore the genus of the involution, is

$$\pi=\frac{2(n-1)(n-2)-2\Sigma v_i\cdot i(i-1)-n(n-2)}{4}.$$

The discussion in § 15 has taken cognizance of all possible positions of singularities on $F(x)=0$. When the singularities of $F(x)=0$ are known the determination of the genus of the involution becomes a matter of numerical calculations.

§ 19. *Cyclic Involutions of Any Order.*

The transition to involutions of any order follows directly, if the involution determines a birational transformation of the curve into itself. If a point P goes into P_1, P_1 into P_2, ... and $P_a=P$ by the transformation, the involution is called *cyclic*. The transformation is always periodic, and is either linear, or the given curve $F(x)=0$ can be put into $(1, 1)$ correspondence with a curve, the equation of which is of the form $f(x_1^a, x_2, x_3)\equiv f^a(x)=0$, and is therefore invariant under the linear transformation:

$$U\equiv\begin{pmatrix} x_1 & x_2 & x_3\\ \theta x_1 & x_2 & x_3\end{pmatrix},\ \theta^a=1.^*$$

If $F(x)=0$ has a γ'_a so does $f^a(x)=0$. The groups of the involution on $f^a(x)=0$ are cut out by a pencil of lines having its vertex at the center of homology $(1, 0, 0)$. If the lines of the pencil cut out only one group, γ'_a is a rational g'_a; if they cut the curve in more than one group the involution is multiple. Hence,

A cyclic involution of order a on a given curve is either rational or, if irrational, is multiple.

A multiple cyclic involution may also be rational.

The genus of γ'_a can be determined in a manner exactly analogous to the one employed in the determination of the genus of an involution of order 2. The singularities of $F(x)=0$, not in exceptional positions, will have to appear in groups of a similar ones giving rise to $1/a$-th, their number of similar singularities on $f(y)=0$. If $F(x)=0$ does not pass through the point $(1,0,0)$, $f(y)=0$ has $\frac{n(a-1)}{2}$ branches touching each other at $(1, 0, 0)$, and so on.

* A. Hurwitz, "Ueber diejenigen ....," *Math. Annalen*, Vol. XXXII.

§ 20. *Non-Cyclic Involutions.*

Given a cone $K_{\psi,\pi}=0$ of order Ψ, genus π and a surface $F_a=0$. The curve C of intersection goes b times through the vertex of the cone and is of order m, genus p, where $m=a\Psi+b$ and p is determined from Sturm's formula *

$$p=a(a-1)\Psi+(a-1)(b-1)+a\pi.$$

The curve C possesses an involution of order a, genus π. Through every point P of C passes a generator of K, which meets C in $a-1$ points besides P. But each generator of K meets but one group of a points, hence K, C are in $(1, a)$ correspondence.

If C is projected on a plane section γ of K, when the center of projection is at the vertex of the cone, C is projected a-fold on γ. If the center of projection 0 is on C but not at the vertex of the cone, C will be projected into a plane curve C_1 of order $m-1$, having $a-1$ branches with a common tangent at 0_1 the point in which the generator of K through 0 pierces the plane of projection. Let P be any point on C. The generator through P and the point 0 determine a plane which passes through the vertex of K, hence cuts it in $\Psi-1$ generators besides the one through 0. The plane meets the plane of projection in a line through 0_1 and contains $\Psi-1$ groups of the involution. We have therefore an illustration of a multiple non-cyclic involution on C_1. The image $\gamma'_{a,\pi}$ may be assumed at will, hence curves having involutions of any order can be constructed, which have a given curve for image of the involution.

The number of coincidences is the number of tangents to C through the vertex of the cone K. By the Cayley-Brill formula this is seen to be $2(a-1+p)$. This, by Castelnuovo's formula already cited (§ 6), is the maximum number an involution $\gamma'_{a,\pi}$ can have.

If the center of projection is on K but not on C the conditions are unchanged except that the vertex of the pencil of lines in the plane of C_1 is now an a-fold point at which all a branches have a common tangent. As before, each line of the pencil contains $\Psi-1$ groups of the involution.

Finally, if the center of projection 0 is not on K, the vertex 0_1 of the plane pencil is not on C_1, and each line of the pencil contains Ψ groups of the involution.

Next, suppose we have a ruled surface $R_{n,\pi}=0$ of order n, and π the genus of a plane section Γ. If $C_m=0$ is a complete intersection of R and $F_k=0$, $m=kn$, its characteristics are connected with those of R by the following relation:

$$m(k+n-2)=r+2d, \tag{10}$$

* R. Sturm, "Ueber das Geschlecht von Curven auf Kegeln," *Math. Annalen*, Vol. XIX (1882), pp. 487–488.

where d is the order of the double curve on R and is given by

$$d=\frac{(n-1)(n-2)}{2}-\pi,$$

and r, the rank of C is $r=2m+2p-2$. Substituting the values of r and d in (10) we get

$$m(k-1)-p-\delta=\frac{k(k-1)n}{2}-k(\pi-1)-1, \tag{11}$$

δ being the number of times R and F touch.

If the complete intersection of R and F is made up of C and s generators, so that $m=kn-s$, we have

$$(m-s)(k+n-2)=r-r'.^*$$

r', the rank of the system of generators, is zero, hence

$$(kn-2s)(k+n-2)=2(kn-s)+2p-2+k(n-1)(n-2)-2k\pi-2s(n-2),$$

or, as above,

$$m(k-1)-p-\delta=\frac{k(k-1)n}{2}-k(\pi-1)-1. \tag{11}$$

If the residual of C_m is another curve $C_{m'}$, and if there exists an $F_{k'}=0$ which cuts R in C_m or $C_{m'}$ and s generators then, by means of relation (11) we can find the genus of C_m or $C_{m'}$ in terms of the characteristics of R, then the genus of the other curve by means of (10), which reduces to the form of (11). Formula (11), which is a generalization of Sturm's formula due to Segre,† is applicable to any curve on a ruled surface.

If now C is projected into the plane curve C_1, the k points in which each generator of R meets C will be projected into a group of k collinear points, but the same line contains $m-k$ other points, not belonging to a group. The lines containing each a group of the involution envelope a curve of class n, birationally equivalent to the dual of Γ. Each of the remaining points on a line, not belonging to the group on that line, belongs to a group on another tangent to the envelope.

In case the surface $F=0$ is also a ruled surface, on C are two distinct involutions, and hence also on C_1. Let $F\equiv R'_{n',\pi'}=0$ and $R_{n,\pi}=0$ have j generators in common, so that $m=nn'-j$, $\delta=2j$, $k=n'$, $k'=n$. From Segre's formula we have

$$(k-1)(nn'-j)-p-2j=\frac{k(k-1)n}{2}-k(\pi-1)-1,$$

$$(n'-1)(nn'-j)-p-2j=\frac{k'(k'-1)n'}{2}-k'(\pi'-1)-1.$$

---

* Salmon, "Analytic Geometry of Three Dimensions," fifth edition, Vol. I (1912), § 346, p. 358.

† C. Segre, "Recherches générales sur les courbes et les surfaces réglées algébriques," *Math. Annalen*, Vol. XXXIV (1889), pp. 1–25.

The number of coincidences in the first involution is $2(n'-1+p)$ and in the second is $2(n-1+p)$. The maximum genus of C when $j=0$ can be obtained from Salmon's theory of postulation.

§ 21. *Restrictions on the Moduli of a Curve Having an Involution.*

The application of the methods used in the preceding pages to the second question we set out to answer is immediate. Let us first consider a simple case. Given an involution of order 2 and genus 1, to find the simplest curve that can possess it; in other words, find the simplest curve upon which a non-singular cubic can be mapped doubly. Consider the cubic

$$y_2^2y_3=\phi_3(y_1, y_3). \tag{12}$$

By means of T^{-1} we find as its image in the A-plane the sextic

$$x_1^2x_3^4=\phi_3(x_1^2, x_3^2). \tag{12'}$$

The sextic has a four-fold point and its genus is therefore 2. We notice that (12′) is invariant not only under L, but also under another homology which replaces x_3 by $-y_3$, or that the existence of one elliptic involution of order 2 on a curve of genus 2 necessitates the existence of another involution of the same order and genus. In the same manner most of the theorems established by Torelli* by transcendental methods, can be proved as simple corollaries of the preceding theorems.

In general, in order to find the genus of a curve which possesses a $\gamma'_{a,\pi}$, we start in the A'-plane with a normal form of a curve of genus π. Every multiple point of the curve, not in an exceptional position, gives rise to a similar multiple point on the image curve in the A-plane. The procedure laid down in § 16 is followed in the determination of correspondents of multiple points in exceptional positions. The genus of the image curve is then easily calculated.

---

* R. Torelli, "Sulle curve di genere due contenenti una involuzione ellitica," *Rend. Acc. Napoli*, s. 3, Vol. XVII (1911), pp. 412–419.

The Set of Eight Self-Associated Points in Space.

By John Rogers Musselman.

Introduction.

Associated point sets were first discussed by Rosanes* and Sturm.† Later a type of self-conjugate association was treated by Study.‡ Recently Coble§ discussed the association of a set P_n^k (n-points in S_k) with a set Q_n^{n-k-2} (n-points in S_{n-k-2}) and derived the complete systems of invariants for P_6^1 and P_6^2.

An interesting case of associated sets occurs when both sets of points are in the same space. The term association as defined implies a mutual ordering of the points. It may be possible to project the one set upon the other in the order of association. The two sets are then said to be self-associated, and the order will be referred to as the identical order. For the P_8^3 this is the well-known set of base points of a net of quadrics. If the associated sets can be projected one upon the other in some order other than the identical, the sets are said to be self-associated in other than the identical order. It is this type of P_8^3 which will be discussed in this paper. In § 2 the general set of eight points self-associated in some order is treated. If this self-association requires that the set be the base points of a net of quadrics, the set is said to be of type B and is discussed in § 3. If the eight points lie on a rational space cubic, the set is of type R and is considered in § 4. Various theorems and facts of immediate use are grouped in § 1. In order to restrict the number of cases, no order of self-association is discussed if in the set two points should coincide, three lie on a line, or four be in a plane.‖ Such a set is said to be of type E.

Those sets of points which can be self-associated in other than the identical order, can naturally be determined only to within projective transformation.

---

* *Crelle*, Bd. LXXXVIII (1880), p. 241.

† *Math. Ann.*, Bd. I (1869), p. 533, and Bd. XXII (1883), p. 569.

‡ *Math. Ann.*, Bd. LX (1905), p. 321.

§ *Transactions*, Vol. XVI (1915), p. 155. This paper hereafter will be cited as C.

‖ This excludes some cases of interest. See footnote on Desmic Systems, p. 81.

Most of the conclusions in § 2 are given in terms of the elliptic parameters of the points; if the curve on them degenerates, the geometrical construction of the set is given. In § 3 the ten types of the self-projective planar quartic given by Wiman* are tabulated and their connection, where possible, shown with specific orders of self-association of the P_8^3. The types of self-associated sets on a rational cubic and the groups connected with them are given in § 4.

§ 1. Let the set of eight points in space be given by the equations

$$(up_1)=0, \quad (up_2)=0, \; \ldots\ldots, \; (up_8)=0.$$

Since any five points in space are linearly related, these equations are connected by four linear relations. Let them be

$$q_{1i}(up_1)+q_{2i}(up_2)+\ldots\ldots+q_{8i}(up_8)\equiv 0, \qquad i=1,\,2,\,3,\,4.$$

Multiplying them respectively by v_i and adding, we have the single identity in u and v,

$$(1) \qquad (vq_1)(up_1)+(vq_2)(up_2)+\ldots\ldots+(vq_8)(up_8)\equiv 0,$$

which leads to the set Q_8^3. Thus a set P_8^3 projectively defines an associated set Q_8^3, and the relation is mutual. If the set P_8^3 is self-associated the above identity can be written as

$$(2) \qquad \lambda_1(up_1)(vq_1)+\lambda_2(up_2)(vq_2)+\ldots\ldots+\lambda_8(up_8)(vq_8)\equiv 0,$$

where the points q_i are merely some permutation of the points p_i. The latter can be replaced by the two following identities:

$$(3) \qquad \begin{aligned} &\lambda_1(up_1)(uq_1)+\lambda_2(up_2)(uq_2)+\ldots\ldots+\lambda_8(up_8)(uq_8)\equiv 0,\\ &\lambda_1(p_1q_1x) \quad +\lambda_2(p_2q_2x) \quad +\ldots\ldots+\lambda_8(p_8q_8x) \quad \equiv 0.\end{aligned}$$

We shall have need of the following facts in the later paragraphs:

(4) If the associated sets P_8^3 and Q_8^3 in S_3 be placed so that the first five points of each constitute the same base, the remaining three points of each set lie in the same plane α, and form polar triangles of the conic Q_α, in α, which is apolar to all sections by α of the basic quadrics.†

(5) If by projection from one of the points q, say q_8, and section by a plane, there is obtained from the remaining seven points $q_1, \ldots, q_7$ a set Q_7^2, this set is associated with the set P_7^2 formed by the points $p_1, \ldots, p_7$ of P_8^3.‡ From this theorem is obtained a result of importance for any order of association containing a cycle of two letters, say (p_7p_8). By projection from p_8 we get seven points p in a plane associated with the set of seven points q. If the

* *Math. Ann.*, Bd. XLVIII (1897), p. 222. † C., p. 159. ‡ C., p. 158.

sets P_8^3 and Q_8^3 are not self-associated, p_7 will have been sent into q_8. Projecting now from q_8 we obtain six points q in a plane associated with six points p in the plane. If the sets P_8^3 and Q_8^3 are self-associated, then p_7 is q_8, and the projected sets in the plane coincide and are self-associated. The set of six self-associated points in the plane has been fully treated by Coble,* and his conclusions are available for our purposes.

(6) Two sets associated with a third set are projective; for the sets are only projectively known so that a third set will projectively define its associated set.

There are twenty-one possible orders in which a P_8^3 may be self-associated, as (12), (12)(34), ..., where the points are indicated by their subscripts and where, for example, by (12345) we mean $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$ is associated with $p_2, p_3, p_4, p_5, p_1, p_6, p_7, p_8$.

§ 2. In this section we treat those orders of self-association of the P_8^3 which permit the eight points to lie on a unique elliptic quartic. Three theorems are now proved which enable us to classify some of the orders of self-association as belonging to types B and E; such types are discussed later.

(7) *Any order of association containing only one cycle of three points is of type E.* Suppose the cycle to be (123)...., where the other five points enter in any possible arrangement save a cycle of three. By applying (6) this order of association (123).... implies projectivity in the order (312)...., and, consequently, is projective in that power of this order which is the least common multiple of the periods of the cycles—not including the cycle of three points. The original order of association is then projective to the order (123), let us say. The transformation defined by this projectivity has multipliers 1, ω, or ω^2; ($\omega^3=1$). Evidently two multipliers at least will be alike and we can choose these two to be unity. There are then three types to consider: 1, 1, 1, ω; 1, 1, ω, ω; 1, 1, ω, ω^2. The transformation having the first set of multipliers has a fixed point and a fixed plane; the one with the second set of multipliers has two lines of fixed points, while the transformation with the third set of multipliers has a line of fixed points and a line of fixed planes. In all these types we can not have five fixed points without four of the P_8^3 lying in a plane and thus giving a P_8^3 of type E. *Hence the following orders* (123), (123)(45), (123)(45)(67), (123)(4567) *and* (123)(45678) *are of type E and will not be discussed in this paper.*

---

* C., p. 163.

(8) *Any order of association containing only one cycle of four points is of type E.* By an argument similar to that in (7) it can be shown that the order of association (1234) .... implies projeetivity to the order of association (13) (24). This type of projective transformation is the well-known harmonic perspectivity in a point and plane, or in two lines. In either case to have four fixed points would require four of the P_8^3 to lie in a plane; and so any order of association containing only one cycle of four points leads to a P_8^3 of type E. *Hence the orders of association* (1234), (1234) (56) *and* (1234) (56) (78) *are of type E and will not be considered in this paper.*

(9) *All further orders of association of odd period are of type B.* In fact all orders of association of odd period are of type B, but as some are included in the statements of the last two paragraphs, we shall say only those orders of odd period, not previously excluded, are of type B. Let Π be the order of association. By (6) the order of association Π implies projectivity to the order of association Π^2. To be of type B, Π is projective to itself in the identical order, or $\Pi^{2n}=\Pi$. This says $\Pi^{2n-1}=1$ which is true if Π is of odd period. Hence all orders of association of odd period are of type B, *and the following orders* (123) (456), (12345) *and* (1234567) *will be reserved for discussion in the next section of this paper.*

(10) The remaining ten orders of self-association will now be taken up in detail, considering first the order (123456) (78). This order implies projectivity to the order (135) (246); thus defining a transformation with multipliers taken from $1, \omega, \omega^2$ $(\omega^3=1)$. Let us give the P_8^3 the following coordinates:

$1: 1,\ 1,\ 1,\ 1,\quad 2: \lambda,\ 1,\ x_3,\ x_4,\quad 3: 1,\ 1,\ \omega^2,\ \omega,\quad 4: \lambda,\ 1,\ \omega^2 x_3,\ \omega x_4,$
$5: 1,\ 1,\ \omega,\ \omega^2,\quad 6: \lambda,\ 1,\ \omega x_3,\ \omega^2 x_4,\quad 7: 1,\ 0,\ 0,\ 0,\quad 8: 0,\ 1,\ 0,\ 0.$

The sixteen equations resulting from the identity

$$\lambda_1(up_1)(vp_2)+\ldots+\lambda_6(up_6)(vp_1)+\lambda_7(up_7)(vp_8)+\lambda_8(up_8)(vp_7)\equiv 0$$

have the following matrix where the row $[ij]$ indicates the coefficients of the equation derived from the coefficient of $u_i v_j$ $(i, j=1, \ldots, 4)$.

$$\begin{array}{lcccccccc}
[11] & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 0 & 0, \\
[22] & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0, \\
[33] & x_3 & \omega^2 x_3 & \omega x_3 & x_3 & \omega^2 x_3 & \omega x_3 & 0 & 0, \\
[44] & x_4 & \omega x_4 & \omega^2 x_4 & x_4 & \omega x_4 & \omega^2 x_4 & 0 & 0, \\
[12] & 1 & \lambda & 1 & \lambda & 1 & \lambda & 1 & 0, \\
[21] & \lambda & 1 & \lambda & 1 & \lambda & 1 & 0 & 1, \\
[13] & x_3 & \lambda\omega^2 & \omega^2 x_3 & \lambda\omega & \omega x_3 & \lambda & 0 & 0, \\
[31] & \lambda & x_3 & \lambda\omega^2 & \omega^2 x_3 & \lambda\omega & \omega x_3 & 0 & 0,
\end{array}$$

are the sides of the reference triangle in the plane $x_1=0$, and the other three are lines through the vertices of the triangle meeting in a point. Moreover, this point is on the quartic curve. For any plane cuts the pencil of quadrics in a pencil of conics on four points of the curve. The conics here are line pairs on the vertices of the reference triangle. They must have another point in common which can not lie on the sides of the triangle, hence the lines through the vertices meet in a point on the curve. We thus obtain four more points on the curve, namely:

$$\begin{array}{lllll} 0, & z_1+z_2, & z_1+z_3, & z_1+z_4, & \alpha \\ z_2+z_1, & 0, & z_2+z_3, & z_2+z_4, & \beta \end{array} \qquad \begin{array}{lllll} z_3+z_1, & z_3+z_2, & 0, & z_3+z_4, & \gamma \\ z_4+z_1, & z_4+z_2, & z_4+z_3, & 0. & \delta \end{array}$$

But 3, 4, α, 5 are in a plane; so also 3, 4, β, 6; 3, 4, γ, 7 and 3, 4, δ, 8.

Calling the elliptic parameters of the points 1, 2, ..., 8, $u_1, u_2, \ldots, u_8$, since 6, 7, 8, α are in a plane,

$$u_3+u_4+u_5\equiv u_6+u_7+u_8, \text{ or } u_3+u_4\equiv -u_5+u_6+u_7+u_8.$$

Similarly, $u_3+u_4\equiv u_5-u_6+u_7+u_8\equiv u_5+u_6-u_7+u_8\equiv u_5+u_6+u_7-u_8,$

whence $\qquad 2(u_3+u_4)=4u_5, \text{ or } -u_3-u_4+2u_5=0.$

Also $\qquad -u_3-u_4+2u_6=0, \quad -u_3-u_4+2u_7=0, \quad -u_3-u_4+2u_8=0.$

These relations say that the four planes on $\overline{-u_3, -u_4}$, tangent to the curve, are tangents at the points u_5, u_6, u_7, u_8. Find that quadric of the pencil having $\overline{u_3, u_4}$ as generator; a plane through this line will cut the quadric in $\overline{-u_3, -u_4}$. The four planes on this line $\overline{-u_3, -u_4}$ tangent to the curve cut out the points u_5, u_6, u_7, u_8. Points 1 and 2 are treated like 3 and 4. Hence $\overline{12}$, $\overline{34}$ are any two generators of the same system of a quadric on the curve; 5, 6, 7, 8 are the points where the curve is tangent to the generators of this same system. The P_8^3 thus contains four absolute constants, and the conditions on the parameters of the points are

$$u_1+u_2=u_3+u_4=2u_5=2u_6=2u_7=2u_8.$$

(13) The second identity for the order of association (12) (34) (56) (78) is

$$(\lambda_1-\lambda_2)(12x)+(\lambda_3-\lambda_4)(34x)+\ldots\ldots+(\lambda_7-\lambda_8)(78x)\equiv 0.$$

If one difference vanishes, say the first, the lines $\overline{34}$, $\overline{56}$, and $\overline{78}$ are on a point, whence four points lie in a plane. If two differences vanish, either two pairs of points coincide or four points lie on a line. If three differences vanish, two points must coincide. These conditions all lead to sets of type E. If none of the differences vanish, the four lines are lines of the same system of generators on a quadric. Moreover, this condition is sufficient because the first

identity can be satisfied by choosing $\lambda_1+\lambda_2=0, \ldots, \ldots, \lambda_7+\lambda_8=0$. This set involves six absolute constants: one for the curve, one for the quadric, and one for each generator. In terms of the elliptic parameters of the points we have

$$u_1+u_2=u_3+u_4=u_5+u_6=u_7+u_8=k.$$

However the case remains for which all the differences may vanish. Applying theorem (5) and projecting from point 7 (or 8) we get six self-associated points in a plane which under the order of association (12) (34) (56) requires the points to lie on three lines meeting in a point.* This means that looking from point 7 (or 8) the projections of the lines $\overline{12}$, $\overline{34}$, $\overline{56}$ upon a plane meet in a point. To do so the line of perspection from 7 (or 8) must meet these lines. The lines of perspection from 7 and 8 are distinct, else four points in a plane. So 7 and 8 lie on two cross generators on the quadric having $\overline{12}$, $\overline{34}$, $\overline{56}$ as generators. Either the four lines are generators of the same system on a quadric, or the quadrics having $\overline{12}$, $\overline{34}$, $\overline{56}$ and $\overline{34}$, $\overline{56}$, $\overline{78}$, respectively, as generators are distinct. Since both are quadrics on the curve, their intersection, $\overline{34}$, $\overline{56}$ and two cross generators, shows 1, 2, 7, 8 must lie on the cross generators whence four points would be in a plane. Hence the two quadrics must be the same one and $\overline{12}$, $\overline{34}$, $\overline{56}$, $\overline{78}$ are generators of the same system on a quadric.

The first identity, $\lambda_1(u1)(u2)+\ldots.+\ldots.+\lambda_7(u7)(u8)\equiv 0$, says that any quadric apolar to the first three pairs must cut $\overline{78}$ harmonically as to 7 and 8. Examine the quadric made of planes $\overline{134}$, $\overline{156}$. The plane $\overline{134}$, being tangent along the generator $\overline{34}$, cuts out on the quadric that generator of the other system through the point 1. But the plane $\overline{156}$ cuts out the same generator. Now the points where these lines cut $\overline{78}$, since the generators coincide, must be at 7 or at 8. In either case, four points lie in a plane, and we are led to a P_8^3 of type E. Therefore the order (12) (34) (56) (78) requires the four lines $\overline{12}$, $\overline{34}$, $\overline{56}$, $\overline{78}$ to be lines of the same system on a quadric. The set contains six constants and is given parametrically by

$$u_1+u_2=u_3+u_4=u_5+u_6=u_7+u_8=k.$$

(14) We can dispose of the order of association (123) (456) (78) in a few words. It implies projectivity to the order (132) (465). It is also associated in the cube of its order, that is, in the order (78). These facts enable us to construct the set. The six points 1, 2, 3, 4, 5, 6 determine a cubic curve on which they lie in two cyclic sets 1, 2, 3 and 4, 5, 6. The transformation sending

* C., p. 161.

these points into each other cyclically has two fixed points on the curve. Join these points, and 7 and 8 are points on the Weddle, determined by the first six points, cut out by this line of fixed points. The quartic curve has degenerated into a cubic and its bisecant, while the P_8^3 contains but one absolute constant.

(15) The order of association (1234) (5678) implies projectivity to the order (13) (24) (57) (68). The points must lie harmonically separated in pairs as to two fixed lines. Let them have the coordinates:

$1: 1, 0, 0, a,\quad 2: 0, 1, b, 0,\quad 3: 1, 0, 0, -a,\quad 4: 0, 1, -b, 0,$
$5: 1, 1, 1, 1,\quad 6: x_1, x_2, x_3, x_4,\quad 7: 1, 1, -1, -1,\quad 8: x_1, x_2, -x_3, -x_4.$

Substituting these values in the identity we obtain sixteen equations which can be satisfied if the following conditions are satisfied:

$$x_4^2+x_1^2=x_3^2+x_2^2=x_1x_2+x_3x_4=x_1x_3-x_2x_4=0,\quad (ax_3-bx_2)^2+(ax_1+bx_4)^2=0.$$

The first four relations are satisfied by $x_1=ix_4$, $x_3=-ix_2\,[i^4=1]$. Putting these values in the remaining condition,

$$(-iax_2-bx_2)^2+(aix_4+bx_4)^2=0,\ \text{or}\ (x_2^2+x_4^2)(ia+b)^2=0,$$

whence either $$x_2^2+x_4^2=0,\ \text{or}\ a=ib.$$

CASE I. $x_1=ix_4$, $x_3=-ix_2$ and $x_2=\pm ix_4$.

If $x_2=ix_4$, letting $x_4=1$, we have $x_1=i$, $x_2=i$, $x_3=1$, $x_4=1$. With these values points 5, 6, 7, 8 are in a plane, so they lead to a P_8^3 of type E.

If $x_2=-ix_4$, letting $x_4=1$, we have $x_1=i$, $x_2=-i$, $x_3=-1$, $x_4=1$, which gives a P_8^3 involving two constants, and whose coordinates are:

$1: 1, 0, 0, a,\quad 2: 0, 1, b, 0,\quad 3: 1, 0, 0, -a,\quad 4: 0, 1, b, 0,$
$5: 1, 1, 1, 1,\quad 6: 1, -1, i, -i,\quad 7: 1, 1, -1, -1,\quad 8: 1, -1, -i, i.$

CASE II. $x_1=ix_4$, $x_3=-ix_2$, $a=ib$ gives a P_8^3 containing two absolute constants, whose coordinates are:

$1: 1, 0, 0, ib,\quad 2: 0, 1, b, 0,\quad 3: 1, 0, 0, -ib,\quad 4: 0, 1, -b, 0,$
$5: 1, 1, 1, 1,\quad 6: 1, -ic, -c, -i,\quad 7: 1, 1, -1, -1,\quad 8: 1, -ic, c, i.$

Both P_8^3's give the lines $\overline{13}$, $\overline{24}$, $\overline{57}$, $\overline{68}$ as generators of the same system of a quadric on the curve.

In Case I the four generators are harmonic, and the double-ratio of the generators through 1, 2, 5, 6 on the edge $\overline{e_1e_4}$ of the reference tetrahedron is

$$\frac{(a-1)(b+i)}{(b-1)(a+i)}.$$

In Case II the double-ratio of $\overline{13}$, $\overline{24}$, $\overline{57}$, $\overline{68}$ is ic, and the double-ratio of the generators, on $\overline{e_1e_4}$, through the points 1, 2, 5, 6 is $\dfrac{(b+i)^2}{b^2-1}$.

(16) The order of association (12345)(67) implies projectivity to the order (13524). Let P_8^3 be

$1: 1,\ 1,\ 1,\ 1,\quad 2: 1,\ \varepsilon^{3a},\ \varepsilon^{3b},\ \varepsilon^{3c},\quad 3: 1,\ \varepsilon^{a},\ \varepsilon^{b},\ \varepsilon^{c},\quad 4: 1,\ \varepsilon^{4a},\ \varepsilon^{4b},\ \varepsilon^{4c},$
$5: 1,\ \varepsilon^{2a},\ \varepsilon^{2b},\ \varepsilon^{2c},\quad 6: 1,\ 0,\ 0,\ 0,\quad 7: 0,\ 1,\ 0,\ 0,\quad 8: 0,\ 0,\ 1,\ 0,$

where a, b, c are all different and different from zero, and ε^a is a fifth root of unity.

The identity (2) for the order of association (12345)(67) is:

$$\lambda_1(u1)(v2)+\ldots.+\lambda_5(u5)(v1)+\lambda_6(u6)(v7)+\ldots.+\lambda_8(u8)(v8)\equiv 0.$$

The coefficients of u_1v_2, u_2v_1, u_3v_3 determine λ_6, λ_7, λ_8, respectively. The remaining coefficients contain only the first five λ's and reduce to eight equations whose matrix is:

$$\begin{array}{lllll}
1 & \varepsilon^{a} & \varepsilon^{2a} & \varepsilon^{3a} & \varepsilon^{4a}, \\
1 & \varepsilon^{c} & \varepsilon^{2c} & \varepsilon^{3c} & \varepsilon^{4c}, \\
1 & 1 & 1 & 1 & 1, \\
1 & \varepsilon^{3b} & \varepsilon^{b} & \varepsilon^{4b} & \varepsilon^{2b}, \\
1 & \varepsilon^{3c} & \varepsilon^{c} & \varepsilon^{4c} & \varepsilon^{2c}, \\
1 & \varepsilon^{3(a+b)} & \varepsilon^{(a+b)} & \varepsilon^{4(a+b)} & \varepsilon^{2(a+b)}, \\
1 & \varepsilon^{3(a+c)} & \varepsilon^{(a+c)} & \varepsilon^{4(a+c)} & \varepsilon^{2(a+c)}, \\
1 & \varepsilon^{3(b+c)} & \varepsilon^{(b+c)} & \varepsilon^{4(b+c)} & \varepsilon^{2(b+c)}.
\end{array}$$

Let ε^d be the remaining fifth root of unity not used in the coordinates of the points. We have two possibilities.

CASE I. $a+b=5$, $c+d=5$, CASE II. $a+c=5$, $b+d=5$,
Iα. $a=1$, $b=4$, $c=2$, $d=3$, IIα. $a=1$, $b=2$, $c=4$, $d=3$,
Iβ. $a=1$, $b=4$, $c=3$, $d=2$. IIβ. $a=1$, $b=3$, $c=4$, $d=2$.

We can always choose $a=1$. If it is not 1, by a proper power of the transformation we can make it 1. Using the above values for a, b, c, d we find the set of eight equations in $\lambda_1, \ldots, \lambda_5$ incapable of solution except for the values in IIβ. Hence the P_8^3 contains no absolute constant and has the following coordinates:

$1: 1,\ 1,\ 1,\ 1,\quad 2: 1,\ \varepsilon^{3},\ \varepsilon^{4},\ \varepsilon^{2},\quad 3: 1,\ \varepsilon,\ \varepsilon^{3},\ \varepsilon^{4},\quad 4: 1,\ \varepsilon^{4},\ \varepsilon^{2},\ \varepsilon,$
$5: 1,\ \varepsilon^{2},\ \varepsilon,\ \varepsilon^{3},\quad 6: 1,\ 0,\ 0,\ 0,\quad 7: 0,\ 1,\ 0,\ 0,\quad 8: 0,\ 0,\ 1,\ 0.$

(17) The order of association (123456) implies projectivity to the order (135)(246). Let the P_8^3 be

$1: 1,\ 1,\ 1,\ 1,\quad 2: \lambda,\ 1,\ x_3,\ x_4,\quad 3: 1,\ 1,\ \omega^2,\ \omega,\quad 4: \lambda,\ 1,\ \omega^2x_3,\ \omega x_4,$
$5: 1,\ 1,\ \omega,\ \omega^2,\quad 6: \lambda,\ 1,\ \omega x_3,\ \omega^2x_4,\quad 7: 1,\ 0,\ 0,\ 0,\quad 8: 0,\ 1,\ 0,\ 0.$

The sixteen equations resulting from the identity are satisfied, for the above choice of coordinates, if $\lambda=-1$, and $x_4=-\omega x_3$. Hence the set contains one absolute constant and can be written as

$$\begin{array}{llll} 1: 1,\ 1,\ 1,\ 1, & 2: -1,\ 1,\ \alpha,\ -\omega\alpha, & 3: 1,\ 1,\ \omega^2,\ \omega, & 4: -1,\ 1,\ \omega^2\alpha,\ -\omega^2\alpha, \\ 5: 1,\ 1,\ \omega,\ \omega^2, & 6: -1,\ 1,\ \omega\alpha,\ -\alpha, & 7: 1,\ 0,\ 0,\ 0, & 8: 0,\ 1,\ 0,\ 0. \end{array}$$

To construct the set choose 7, 8, H_1, H_2 as the reference points, and 1 as unit point. Let H_3 be the point on $\overline{78}$ cut out by the plane $\overline{1H_1H_2}$ and P be the fourth harmonic on $\overline{78}$ of H_3 as to 7 and 8. Then 3 and 5 are determined as those points which with 1 have $H_1H_2H_3$ as their Hessian triangle. Select 4′ in the plane $\overline{H_1H_2H_3}$ as any point on the harmonic line of $\overline{H_31}$ as to $\overline{H_3H_1}$ and $\overline{H_3H_2}$; likewise determining 5′ and 6′ as those points which with 4′ have $H_1H_2H_3$ as their Hessian triangle. Projecting from 7 upon $\overline{PH_1H_2}$ the points 4′, 5′, 6′ project into 4, 5, 6, whence the set is determined completely. By changing the sign of α, we get a set 4″, 5″, 6″ in $\overline{H_1H_2H_3}$, which projected from 8 gives 4, 5, 6 in the plane $\overline{PH_1H_2}$. The set of points lie on a cubic through 1, 2, ..., 6 and a bisecant on which lie 7 and 8. Hence the order of association (123456) is possible and the set of points contains one absolute constant.

(18) The order of association (12345678) implies projectivity to the order (1357) (2468). Let the P_8^3 be

$$\begin{array}{lll} 1: 1,\ 1,\ 1,\ 1, & 2: x_1,\ x_2,\ x_3,\ x_4, & 3: 1,\ i,\ -1,\ -i, \\ 4: x_1,\ ix_2,\ -x_3,\ -ix_4, & 5: 1,\ -1,\ 1,\ -1, & 6: x_1,\ -x_2,\ x_3,\ -x_4, \\ & 7: 1,\ -i,\ -1,\ i, & 8: x_1,\ -ix_2,\ -x_3,\ ix_4. \end{array}$$

The sixteen equations resulting from substituting these values in the identity are satisfied if we choose x_1, x_2, x_3, x_4 to satisfy $i(x_1^2-x_3^2)-(x_2^2-x_4^2)=0$, whence the set contains two absolute constants. The quadric passes through 2 and 6, and has $\overline{13}$, $\overline{57}$, $\overline{15}$, $\overline{37}$ as generators. This apparent lack of symmetry is explainable. In the order (12345678) we pick out the points 2 and 6, the numbers enclosing them are 1, 3, 5, 7, hence we use for generators besides $\overline{15}$, $\overline{37}$, the lines $\overline{13}$, $\overline{57}$. Similarly, if we isolate points 4 and 8, the numbers enclosing them are 3, 5, 1, 7; hence there is a quadric having $\overline{15}$, $\overline{37}$, $\overline{35}$, $\overline{17}$ as generators, and on 4 and 8, namely:

$$i(x_1^2-x_3^2)+(x_2^2-x_4^2)=0.$$

If we choose $x_4=1$, $x_3=b$, $x_2=a$, then $x_1=\sqrt{b^2+i(a^2-1)}$. Hence the order of association (12345678) is possible, the set contains two absolute constants, lies on an elliptic quartic, and its coordinates are given above.

(19) For the order of association (12) (34) (56) the identities are satisfied if we take points 5, 6, 7, 8 as the reference points, 4 as unit point, and 1, 2, 3 as x, y, z, respectively, by $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$, $\lambda_5=\lambda_6$, and if the following equations are satisfied,

$$\begin{aligned} x_2y_3+\ x_3y_2+x_1y_1-x_2y_2-x_1y_3-x_3y_1&=0,\\ x_2y_4+\ x_4y_2+x_1y_1-x_2y_2-x_1y_4+x_4y_1&=0,\\ x_3y_4+x_4y_3+2x_1y_1-x_1y_3-x_3y_1-x_1y_4-x_4y_1&=0. \end{aligned}$$

Solving for y in terms of x we note:

$$y_1=y_2=y_3=y_4=x_1-x_2[2x_3x_4+2x_1x_2-x_2x_4-x_2x_3-x_1x_3-x_1x_4].$$

Now the four values of y_i can not be equal, else this point would coincide with 4, neither can $x_1=x_2$, or four points would be in a plane, whence point 1 must lie on the above quadric, and then point 2 will lie on a line. Point 3 is determined by the conditions $z_1:z_2:z_3:z_4=x_1y_1:x_2y_2:x_1y_3+x_3y_1-x_1y_1:x_1y_4+x_4y_1-x_1y_1$. Hence this P_8^3 contains three absolute constants. If we choose $x_1=1$, $x_2=a$, $x_3=b$, then $x_4=\dfrac{b+ab-2a}{2b-a-1}$.

The coordinates of point 2 are

$$y_1=1+\lambda,\quad y_2=\frac{b-1}{b-a}(1-\lambda),\quad y_3=\frac{2\lambda(b-1)}{a-1},\quad y_4=\frac{2(b-1)}{2b-a-1};$$

while point 3 is

$$z_1=1+\lambda,\quad z_2=\frac{a}{b-a}[(b-1)(1-\lambda)],\quad z_3=\frac{b-1}{a-1}[2\lambda+(1+\lambda)(a-1)],$$
$$z_4=\frac{(b-1)[2+(1+\lambda)(a-1)]}{2b-a-1}.$$

Therefore the order of association (12) (34) (56) is possible; the set contains three absolute constants and its coordinates are given above.

§ 3. In this section we shall treat the P_8^3 when they are the base points of a net of quadrics. As such they are self-associated in the identical order, and if self-associated in some given order they are therefore projective in that order. The connection between the general planar quartic (genus 3) and the net of cubic curves on seven points in the plane is well known. The ∞^2 elliptic quartics on the eight base points project from one of those points into a net of cubics on seven points in the plane. Hence an intimate relation exists between the eight base points and the planar quartic. If now the P_8^3 is self-associated in some order and thereby projective in that order, we get a birational transformation of the planar quartic into itself, which is, moreover, a collineation. The types of self-projective quartics have been tabulated by Wiman, and in

this section we shall, where possible, connect each type with an order of self-association. Since the planar quartic is not of genus 3, if two points coincide, three lie on a line, or four be in a plane,* no orders of self-association will enter here which were of type E and which were not discussed in the preceding section.

Associated with the planar quartic are its thirty-six systems of contact cubics. If a collineation permutes some of these systems leaving at least one system fixed, there will be a self-association corresponding to it of the P_8^3. If, however, all of these thirty-six systems are permuted under the collineation, then we can not connect an order of self-association with the quartic. The twenty-eight double-tangents of the quartic shall be designated by the twenty-eight symbols [12], ..., [78], thus furnishing at a glance the number of double-tangents fixed under an order of association.

The ten types of self-projective quartics are:

1°	$x_3^4+x_3^2f_2(x_1, x_2)+f_4(x_1, x_2)=0,$	6°	$x_3^3x_1+x_1^4+x_1^2x_2^2+x_2^4=0,$
2°	$x_3^3f_1(x_1, x_2)+f_4(x_1, x_2)=0,$	7°	$x_3^3x_2+x_2^3x_1+x_1^3x_3=0,$
3°	$ax_3^2x_2^2+bx_3x_2x_1^2+x_3^3x_1+x_2^3x_1+x_1^4=0,$†	8°	$x_3^4+x_1^3x_2+x_1x_2^3=0,$
4°	$x_3^4+f_4(x_1, x_2)=0,$	9°	$x_3^3x_1+x_1^3x_2+x_2^4=0,$
5°	$x_3^4+ax_3^2x_1x_2+x_1^4+bx_1^2x_2^2+x_2^4=0,$†	10°	$x_3^4+x_2^4+x_1x_2^3=0.$

Quartic 1° has four fixed double-tangents. This excludes the orders of association (12) and (12) (34) as they both have more fixed lines; leaving the orders (12) (34) (56) and (12) (34) (56) (78) to be considered. Apply theorem (5) to the order (12) (34) (56). The order of association (12) (34) on the six projected points in a plane requires (a) the six points to lie on a conic, or (b) four on a line, or (c) six on a line.‡ The last two conditions would require the P_8^3 to have four points at least in a plane, while (a) makes the points lie on a quadric cone. If this happens the planar quartic associated with the P_8^3 has a double-point and is no longer of genus 3. Hence, if we can connect any order of association with quartic 1°, it must be the order (12) (34) (56) (78). This we can do, and the conditions on the elliptic parameters of the points are $u_1+u_2=u_3+u_4=u_5+u_6=u_7+u_8=p/4$ (p=period). This set contains four absolute constants; the modulus is not an absolute constant, and this necessarily is the number of constants in the equation 1°.

---

* This excludes the P_8^3 which form two desmic tetrahedra, which set is unaltered by a G_{192}. Of course this P_8^3 is of type B, and the planar quartic associated with it is the complete quadrilateral.

† Types given by Wiman are $x_3^2x_2^2+x_3x_2x_1^2+x_3^3x_1+x_2^3x_1+x_1^4=0$, and $x_3^4+x_3^2x_1x_2+x_1^4+x_1^2x_2^2+x_2^4=0$. The most general quartics of these types contain two constants, and are as given above.

‡ C., p. 161.

Quartic 2° is invariant under a perspective G_3 and has $x_3=0$ as a fixed undulation tangent. The group associated with quartic 3° is a non-perspective G_6 with $x_1=0$ as a fixed double-tangent. Of the two orders of association of period 3, we discard (123), since it has too many fixed lines, and study (123)(456). The P_8^3 is associated and projective in the order (123)(456). The identity can be satisfied if the set has the following coordinates:

$$\begin{array}{llll} 1:\ 1,\ 1,\ 1,\ 1, & 2:\ 1,\ 1,\ \omega^2,\ \omega, & 3:\ 1,\ 1,\ \omega,\ \omega^2, & 4:\ ab,\ 1,\ a,\ b, \\ 5:\ ab,\ 1,\ \omega^2 a,\ \omega b, & 6:\ ab,\ 1,\ \omega a,\ \omega^2 b, & 7:\ 1,\ 0,\ 0,\ 0, & 8:\ 0,\ 1,\ 0,\ 0. \end{array}$$

The set contains two absolute constants, and lies on a cubic curve and its bisecant. Given the curve and points 1, 2, 3 on it, they determine a cyclic transformation of period 3. Choose 4 as any point on the curve, then 5 and 6 are determined as those points which with 4 form a cyclic set under the transformation. The two fixed points of the transformation are then known. Draw their join, and 7 and 8 are a pair of points on this line harmonic to the Weddle points on the line. Thus the set can be constructed.

Both quartics 2° and 3° contain two constants and have one tangent line fixed, but quartic 3° is invariant under a dihedral $G_{2.3}$. Whether or not the P_8^3 is invariant under a $G_{2,3}$ will decide the question as to with which quadric shall be connected the order of association (123) (456). Since we have just seen that the order of association is possible, one at least of the thirty-six systems of contact cubics is fixed. The quartic, being invariant under a G_3, must have two other systems of contact cubics fixed, and we can represent them by $\overline{1237}$, $\overline{4568}$ and $\overline{1238}$, $\overline{4567}$. If the P_8^3 is invariant under a $G_{2,3}$ the transformation of period 2 can not be a harmonic perspectivity in a point and plane—else four points in a plane—and is consequently a harmonic perspectivity in two fixed lines. Moreover, it must be of the type $(ab)(cd)(ef)(gh)$ for the P_8^3, if projective in an order of period 2, is also self-associated in that order, and we saw that the above-mentioned type of period 2 is the only one that exists.

This transformation must send (123)(456) into its inverse, leave $\overline{1237}$, $\overline{4568}$ and $\overline{1238}$, $\overline{4567}$ unaltered; consequently is (14)(26)(35)(78), or (15)(24)(36)(78), or (16)(25)(34)(78). But the P_8^3 is not invariant under any one of these three transformations. Hence we conclude that the order of association (123)(456) is to be connected with quartic 2° and none can be connected with quartic 3°.

Quartic 4° has four fixed undulation tangents, while quartic 5° has none. The orders of association (1234), (1234)(56), and (1234)(56)(78) are excluded since they do not have the same number of fixed lines. The remain-

ing order of period 4 has no fixed lines, hence we connect (1234) (5678) with quartic 5°. The set of points is therefore projective and self-associated in the order (1234) (5678). Let them have the coordinates:

$$\begin{array}{llll} 1:\ 1,\ 1,\ 1,\ 1, & 2:\ 1,\ i,\ -1,\ -i, & 3:\ 1,\ -1,\ 1,\ -1, \\ 4:\ 1,\ -i,\ -1,\ i, & 5:\ x_1,\ x_2,\ x_3,\ x_4, & 6:\ x_1,\ ix_2,\ -x_3,\ -ix_4, \\ & 7:\ x_1,\ -x_2,\ x_3,\ -x_4, & 8:\ x_1,\ -ix_2,\ -x_3,\ ix_4, \end{array}$$

The sixteen equations resulting from the identity are reducible to six in $\lambda_5, \lambda_6, \lambda_7, \lambda_8$, namely:

$$\begin{array}{ll} (x_1^2-\ x_3^2)(\lambda_5+\lambda_6+\lambda_7+\lambda_8)=0, & (x_2^2\ -x_1x_3)(\lambda_5-\ \lambda_6+\lambda_7-\ \lambda_8)=0, \\ (x_1^2-x_2x_4)(\lambda_5+\lambda_6+\lambda_7+\lambda_8)=0, & (x_1x_4-x_2x_3)(\lambda_5-i\lambda_6-\lambda_7+i\lambda_8)=0, \\ (x_2^2-\ x_4^2)(\lambda_5-\lambda_6+\lambda_7-\lambda_8)=0, & (x_1x_2-x_3x_4)(\lambda_5+i\lambda_6-\lambda_7-i\lambda_8)=0. \end{array}$$

For these equations to be consistent, one at least of the multipliers must vanish. If $x_1^2-x_3^2=x_1^2-x_2x_4=0$, let $x_1=1$, $x_2=a$, then $x_3=\pm 1$; $x_4=1/a$. Using $x_3=1$, points 1, 3, 5, 7 lie in a plane, while if $x_3=-1$, points 1, 3, 6, 8 lie in a plane, so the above hypothesis is impossible. A similar argument shows that if $x_2^2-x_4^2=x_2^2-x_1x_3=0$, four points will lie in a plane, so this assumption is likewise untenable.

If $x_1x_4-x_2x_3=0$ by letting $x_1=1$, $x_2=a$, $x_3=b$, then $x_4=ab$ and the P_8^3 is non-degenerate, lying on a quartic curve. Its coordinates are:

$$\begin{array}{llll} 1:1,\ 1,\ 1,\ 1, & 2:1,\ i,\ -1,\ -i, & 3:1,\ -1,\ 1,\ -1, & 4:1,\ -i,\ -1,\ i, \\ 5:1,\ a,\ b,\ ab, & 6:1,\ ia,\ -b,\ -iab, & 7:1,\ -a,\ b,\ -ab, & 8:1,\ -ia,\ -b,\ iab. \end{array}$$

The four lines $\overline{13}$, $\overline{24}$, $\overline{57}$, $\overline{68}$ are generators of a quadric on the curve. The points where these lines cut the line $\overline{e_2e_4}$ of the reference tetrahedron are, respectively, 0, 1, 0, 1; 0, 1, 0, −1; 0, 1, 0, b; 0, 1, 0, $-b$. Calling them, respectively, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ the double ratio on $\overline{e_2e_4}$ of $\{\alpha_1\alpha_2\alpha_3\alpha_4\}$ is $\left(\frac{1-b}{1+b}\right)^2$.

The cross generators of the quadric through the points 1, 3, 2, 4 cut the side $\overline{e_3e_4}$ of the reference tetrahedron in four points whose double ratio is −1. Hence these cross generators are harmonic. The set contains two absolute constants.

If we assume $x_1x_2-x_3x_4=0$, we get a set of points projectively equivalent to the set just discussed, giving no new types. If we assume that more than one of the multipliers of the equations in $\lambda_5, \lambda_6, \lambda_7, \lambda_8$ vanish, then either two points coincide or four points lie in a plane. Hence the order of association (1234) (5678) is connected with quartic 5°, P_8^3 lies on an elliptic quartic and contains two absolute constants.

To quartic 4° we can connect no order of self-association.

Quartic 6° is invariant under a cyclic G_6 whose square $(1, 1, j)$ is connected with the order of association (123) (456) and whose cube $(1, 1, -1)$ with the order (12) (34) (56) (78). In examining the orders of association of period 6 we note that (123) (45) and (123) (45) (67) have not the right number of fixed lines, one. The cube of the order (123456) contains only three cycles of two numbers, while the cube of (123) (456) (78) contains one cycle of two numbers, so neither can be used. The remaining order (123456) (78) is of type E, hence the quartic would not be the general one. Therefore we can not connect an order of self-association with the quartic 6°.

Quartic 7° is the well-known quartic of Klein,* invariant under a G_{168}. With it we connect the order of association (1234567) if it exists. Let us then, with Klein, take the coordinates of the set to be:

$$\begin{array}{lll} 1:\ 1,\ \ 2,\ \ 2,\ \ 2, & 2:\ 1,\ 2\epsilon^{-1},\ 2\epsilon^{-2},\ 2\epsilon^{-4}, & 3:\ 1,\ 2\epsilon^{-2},\ 2\epsilon^{-4},\ 2\epsilon^{-1}, \\ 4:\ 1,\ 2\epsilon^{-3},\ 2\epsilon^{-6},\ 2\epsilon^{-5}, & 5:\ 1,\ 2\epsilon^{-4},\ 2\epsilon^{-1},\ 2\epsilon^{-2}, & 6:\ 1,\ 2\epsilon^{-5},\ 2\epsilon^{-3},\ 2\epsilon^{-6}, \\ & 7:\ 1,\ 2\epsilon^{-6},\ 2\epsilon^{-5},\ 2\epsilon^{-3}, & 8:\ 1,\ \ 0,\ \ 0,\ \ 0, \end{array}$$

where $\epsilon^7=1$.

The sixteen equations resulting from substituting these values in the identity are satisfied by choosing $\lambda_1=\lambda_2=\ldots=\lambda_7=1$. Thus the set exists, contains no absolute constant and is connected with quartic 7°. With the above coordinates the transformations S and T of Klein are, respectively, (1234567) and (18) (27) (34) (56). Not only is this set self-associated in the identical order and the order (1234567) but in one hundred and sixty-six others. By using S and T we can easily find them and learn they are of four types: forty-eight of type (178) (246), fifty-six of type (1358) (2674), twenty-one of type (18) (27) (34) (56), forty-two of type (1234567), which, with the identity, make the one hundred and sixty-eight.

The multipliers of the group leaving 8° unaltered are $1, -1, \frac{1+i}{2}$. The square of this is $1, 1, i$; the group leaving 4° unaltered. There is no order of self-association connected with 4° and, consequently, none can be connected with quartic 8°.

No order of association of period nine appears among those of our list, so we have none to connect with quartic 9°.

Quartic 10° is invariant under a G_{12}. The only transformation of period 12 is (1234) (567) which we saw was of type E. Therefore quartic 10° cannot be connected with an order of association.

---

* Klein, "Elliptischen Modulfunkturiem," Bd. I, pp. 701, 724–725.

Hence only four of the ten types of self-projective quartics can be connected with an order of self-association of the P_8^3, namely:

1° with (12)(34)(56)(78) 5° with (1234)(5678),

2° with (123)(456), 7° with (1234567).

§ 4. If the P_8^3 lies on a rational cubic, it is self-associated in the identical order. Excluding all of type E we find that the P_8^3 on a rational cubic is projective and hence self-associated in the following orders: (12)(34)(56), (12)(34)(56)(78), (123)(456), (1234)(5678), (123456), (1234567), and (12345678). The binary octavics giving the parameters of the P_8^3 for each order of self-association are, respectively:

$$(x_1^2+x_2^2)(x_1^2+\alpha x_2^2)(x_1^2+\beta x_2^2)x_1x_2=0, \qquad (x_1^4+x_2^4)(x_1^4+\alpha x_2^4)=0,$$

$$(x_1^2+x_2^2)(x_1^2+\alpha x_1^2)(x_1^2+\beta x_2^2)(x_1^2+\gamma x_2^2)=0, \qquad (x_1^6+x_2^6)x_1x_2=0,$$

$$(x_1^3+x_2^3)(x_1^3+\alpha x_2^3)x_1x_2=0, \qquad (x_1^7+x_2^7)x_1=0, \quad (x_1^8+x_2^8)=0.$$

The number of absolute constants for each P_8^3 is obvious from the above octavics.

In § 2 and § 3 are given the possible types of self-associated sets. A problem that suggests itself for the future is to determine those P_8^3's which are self-associated in more than one order, and the groups connected with them. This was done for the order (1234567) in § 3, and the one hundred and sixty-eight ways of self-association discussed. We shall do the same now for the P_8^3 on a rational cubic.

In studying the dihedral groups connected with the above P_8^3 we need look for a $G_{2,n}$ only for $n<5$ if n is odd. If $n \geqq 5$, $2n>8$, and with the odd integer n we could use only n again, which likewise makes a number greater than 8. Hence for n odd $n \leqq 3$. Similarly for n even $n \leqq 8$. Therefore the possible G_{2n}'s must be found where $n=2, 3, 4, 6, 8$. In the accompanying table are given the numbers for each value of n.

2	3	4	6	8
2	2	2	2	2
2	3	4	6	8
2	3	4	6	8
4	6	8	12	16

The $G_{2,2}$ has two types	$(x_1^2+x_2^2)(x_1^2+\alpha x_2^2)(\alpha x_1^2+x_2^2)x_1x_2$,
	$(x_1^4+\alpha x_1^2x_2^2+x_2^4)(x_1^4+\beta x_1^2x_2^2+x_2^4)$.
The $G_{2,3}$ has two types	$(x_1^6-x_2^6)x_1x_2$, $(x_1^6+\alpha x_1^3x_2^3+x_2^6)x_1x_2$.
The $G_{2,4}$ has two types	$(x_1^8-x_2^8)$, $(x_1^8+\alpha x_1^4x_2^4+x_2^8)$.
The $G_{2,6}$ has one type	$(x_1^6+x_2^6)x_1x_2$.
The $G_{2,8}$ has one type	$x_1^8+x_2^8$.
The tetrahedral G_4 is	$x_1^8-14x_1^4x_2^4+x_2^8$.

Conclusion.

The general P_8^3 on an elliptic quartic can be self-associated in the following orders: (12), (12)(34), (12)(34)(56), (12)(34)(56)(78), (123)(456)(78), (1234)(5678), (12345)(67), (123456) and (12345678).

The P_8^3, which is the base points of a net of quadrics, can be self-associated in the following orders, besides the identity: (12)(34)(56)(78), (123)(456), (1234)(5678) and (1234567). To each of these we have connected a particular planar quartic (genus 3), which is self-projective.

The P_8^3 on a rational cubic can be self-associated in the following orders: (12)(34)(56), (12)(34)(56)(78), (123)(456), (1234)(5678), (123456), (1234567) and (12345678). The groups connected with sets are also given. The discussion throughout the paper was restricted to sets of points, of which no two coincide, no three lie on a line, no four in a plane.

Associate Minimal Surfaces.

BY JAMES K. WHITTEMORE.

It is a well-known fact, first proved by Schwarz,* that corresponding points of a family of associate minimal surfaces, corresponding in a sense presently to be explained, lie on an ellipse. In Part I of this paper we find the locus of the extremities of these ellipses, which we call Schwarz's ellipses, then find the envelope of a family of associate minimal surfaces and prove that the latter coincides with part of the locus named for real minimal surfaces applicable to surfaces of revolution, with certain exceptions; in Part II it is shown that this coincidence, with coincidence of corresponding points, occurs only when the minimal surfaces, supposed real, are applicable to a surface of revolution.

The Enneper-Weierstrass equations of a minimal surface S are

$$\left.\begin{aligned} x&=\frac{1}{2}\int(1-u^2)F(u)\,du+\frac{1}{2}\int(1-v^2)\phi(v)\,dv=U_1+V_1,\\ y&=\frac{i}{2}\int(1+u^2)F(u)\,du-\frac{i}{2}\int(1+v^2)\phi(v)\,dv=U_2+V_2,\\ z&=\int uF(u)\,du+\int v\phi(v)\,dv=U_3+V_3.\end{aligned}\right\}\tag{1}$$

The parameters u, v are the parameters of the minimal curves of S. When S is real F and ϕ are conjugate functions, and for a real point with a real tangent plane u, v have conjugate values.† The *adjoint* surface has the equations

$$x_1=i(U_1-V_1),\quad y_1=i(U_2-V_2),\quad z_1=i(U_3-V_3).$$

The equations of the associate minimal surface S_α are

$$\left.\begin{aligned} x_\alpha&=U_1e^{i\alpha}+V_1e^{-i\alpha}=x\cos\alpha+x_1\sin\alpha,\\ y_\alpha&=U_2e^{i\alpha}+V_2e^{-i\alpha}=y\cos\alpha+y_1\sin\alpha,\\ z_\alpha&=U_3e^{i\alpha}+V_3e^{-i\alpha}=z\cos\alpha+z_1\sin\alpha.\end{aligned}\right\}\tag{2}$$

Values of α differing by $\pi/2$, substituted in (2), give adjoint surfaces; corresponding points of associate minimal surfaces are given by (2) when u, v are

---

* H. A. Schwarz, "Miscellen aus dem Gebiete der Minimalflächen," *Journal de Crelle*, Vol. LXXX (1875).

† Eisenhart, "Differential Geometry," p. 256.

fixed and α varies. The locus of $x_\alpha, y_\alpha, z_\alpha$ for fixed u, v is Schwarz's ellipse. Points of the ellipse on adjoint surfaces are the extremities of conjugate diameters.*

I.

§1. **The Locus L.** To find the extremities of the principal axes of the ellipse (2) we determine a point (α) such that the tangent at this point is perpendicular to the line joining it with the center. This condition gives

$$\Sigma(x\cos\alpha+x_1\sin\alpha)(-x\sin\alpha+x_1\cos\alpha)=0,$$

from which

$$\tan 2\alpha=\frac{2\Sigma xx_1}{\Sigma(x^2-x_1^2)}=i\,\frac{\Sigma(U_1^2-V_1^2)}{\Sigma(U_1^2+V_1^2)}.$$

The last equation determines two values of α, differing by $\pi/2$, unless $\Sigma U_1^2=\Sigma V_1^2=0$. The vanishing of these two sums is the condition that the ellipse be a circle. It may easily be proved that if Schwarz's ellipse is a circle for all points of a real minimal surface the latter is a plane. We find

$$\cos 2\alpha=\pm\frac{\Sigma(U_1^2+V_1^2)}{2\sqrt{\Sigma U_1^2\Sigma V_1^2}}.$$

Choosing the upper sign,

$$\cos\alpha=\pm\frac{1}{2}\,\frac{\sqrt{\Sigma U_1^2}+\sqrt{\Sigma V_1^2}}{\sqrt[4]{\Sigma U_1^2\Sigma V_1^2}},\quad \sin\alpha=\pm\frac{i}{2}\,\frac{\sqrt{\Sigma U_1^2}-\sqrt{\Sigma V_1^2}}{\sqrt[4]{\Sigma U_1^2\Sigma V_1^2}}.$$

Choosing again the upper signs, we have for one vertex, from (2),

$$x=U_1\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}+V_1\sqrt[4]{\frac{\Sigma U_1^2}{\Sigma V_1^2}},\tag{3}$$

with similar expressions for y and z. The two radicals in (3) are reciprocals, and conjugate for conjugate u, v. The other combinations of signs in the preceding equations give the other vertices of the ellipse; it appears that the four vertices are given by (3) by the four different determinations of the first radical. The locus L is the locus of the four vertices of Schwarz's ellipses, and its equations, in the parameters u, v, are given by (3) with the similar equations for y and z. The locus consists evidently of four nappes, symmetrical in pairs with respect to the origin; these symmetrical pairs we call L_1, L_2 and L_3, L_4. The squares of the semi-axes of the ellipse are given by $2(\Sigma U_1V_1\pm\sqrt{\Sigma U_1^2\Sigma V_1^2})$.

* Scheffers, "Einführung in die Theorie der Flächen," 2d ed., p. 362.

Comparing equations (2) and (3) it is evident that the curve $\sqrt[4]{\Sigma V_1^2/\Sigma U_1^2}=e^{i\alpha}$ is common to the two surfaces S_α and L_1; similarly, curves given by

$$\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=-e^{i\alpha},\quad \sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=-ie^{i\alpha},\quad \sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=ie^{i\alpha},$$

are common, respectively, to S_α and L_2, S_α and L_3, S_α and L_4; it is to be noted that the first of the last three equations, for example, is also the equation of a curve common to $S_{\alpha+\pi}$ and L_1.

It may be proved that a part of the locus L coincides with one of the associate minimal surfaces only if the latter are plane.

§ 2. **The Envelope of S_α.** If we substitute in (2) the value of α in terms of u, v taken from the equation $|\partial x/\partial u\ \partial y/\partial v\ \partial z/\partial \alpha|=0$, we should expect to find both the envelope and the locus of singular points of the surfaces S_α, but it appears that only the envelope results, for the equation is an identity in α at the singular points. For the envelope it gives

$$e^{i\alpha}=\pm\sqrt{\frac{(u+v)V_1+i(v-u)V_2+(uv-1)V_3}{(u+v)U_1+i(v-u)U_2+(uv-1)U_3}}.$$

Evidently the envelope consists of two nappes symmetrical with respect to the origin.

§ 3. **Minimal Surfaces Applicable to Surfaces of Revolution.** We apply the results of the preceding sections to families of real associate minimal surfaces applicable to surfaces of revolution.* All such surfaces are given by (1) where,

$$F(u)=cu^{m-2},\quad \phi(v)=\bar{c}v^{m-2},$$

m being a real constant, c and $\bar{c}$ conjugate constants. We call such surfaces B surfaces, as they were discovered by E. Bour, in particular a surface for any special value of m, not zero, B_m. It has been proved that all the surfaces associate to B_m are congruent, so that B_m is defined except for a homothetic transformation, and we may without restriction suppose c real; the associates of B_m are obtained by rotating the latter about the Z-axis; B_m and B_{-m} are congruent, so that we may suppose m positive; the curves of the surface, v/u constant, are geodesics and correspond to the meridians of the applicable surface of revolution; the curves, uv constant, are curves of constant total curvature, and correspond to the parallels of the surface of revolution. It is easily shown that the curves of B_m

$$\left(\frac{v}{u}\right)^m=1,\quad \left(\frac{v}{u}\right)^m=-1,$$

* We have given an account of these surfaces and of the literature concerning them in a paper published in the *Annals of Mathematics*, Second Series, Vol. XIX, No. 1, September, 1917.

are respectively lines of curvature and asymptotic lines, and that the surface, when the constants of integration are taken as zero, is cut by the xy-plane in the latter. The value $m=0$ gives the minimal helicoids including the catenoid and the right helicoid; $m=2$ gives Enneper's surface.

Excluding $m=0, 1$, and taking all constants of integration as zero, we find

$$\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=\sqrt{\frac{(u+v)V_1+i(v-u)V_2+(uv-1)V_3}{(u+v)U_1+i(v-u)U_2+(uv-1)U_3}}=\left(\frac{v}{u}\right)^{m/2}.$$

It follows that, for B_m, L_1 and L_2 coincide with the envelope of the associate surfaces. It is readily shown that L_1 is a surface of revolution whose axis is the Z-axis, and that L_3 is the xy-plane λ. If $m>1$, L_1 and L_2 form the locus of the extremities of the minor axes of the Schwarz ellipses, while L_3 and L_4, coinciding with the xy-plane, contain the major axes; if $m<1$, the situation is reversed. All ellipses corresponding to points of a curve of constant total curvature of B_m are equal. The curve previously mentioned, common to B_m and L_1 is $(v/u)^{m/2}=1$, which gives $y/x=\tan(2k\pi/m)$ where k is any integer; for B_m and L_2, $(v/u)^{m/2}=-1$ giving $y/x=\tan[(2k+1)\pi/m]$. The curves on L_1 are congruent plane curves, being meridians of the surface of revolution; they are lines of curvature and also geodesics of B_m, which is tangent to L_1 along each curve. The plane of each curve is a plane of symmetry of B_m. Similar statements apply to B_m and L_2. The curves common to B_m and L_3; B_m and L_4 are given respectively by

$$\left(\frac{v}{u}\right)^{m/2}=-i;\quad \frac{y}{x}=-\cot\frac{(2k+\frac{1}{2})\pi}{m},\quad z=0,$$
$$\left(\frac{v}{u}\right)^{m/2}=\quad i;\quad \frac{y}{x}=-\cot\frac{(2k-\frac{1}{2})\pi}{m},\quad z=0,$$

and are straight lines in the xy-plane. Each line is a line of symmetry of B_m.

From the previous paragraph follow several theorems first proved by Ribaucour. A surface B_m, $|m|\neq 1$, has a number of congruent plane geodesic lines of curvature lying in planes which contain the Z-axis; half or all of these, depending on the value of m, are obtained by rotating one of them through successive angles $2\pi/m$, and an equal number are obtained by rotating about the Z-axis the symmetry of one of these curves with respect to the xy-plane through the angle π/m, then through successive angles $2\pi/m$; straight asymptotic lines of the surface, lying in the xy-plane, bisect the angles of the planes of the lines of curvature named above; if $m=p/q$, where p and q are integers with no common factor, and $q=1$ if m is an integer, the number of these plane lines of curvature is p and is equal to the number of straight

asymptotic lines referred to; if m is irrational the number of each of these sorts of lines is infinite. Ribaucour's results are not so fully stated nor are they completely proved.

II.

§ 4. **Formulation of the Converse Problem.** The remainder of this paper is devoted to the investigation of the question: For what real minimal surfaces does the envelope of the associate surfaces coincide with part of the locus L, with coincidence of points given by the same u, v? We prove that the only surfaces having this property are the plane and B_m, $|m| \neq 1$, so that this coincidence is a characteristic property of real minimal surfaces applicable to surfaces of revolution. It must, however, be remarked that it does not hold for B_1 or for the minimal helicoids, $m=0$.

For the required coincidence we must have, for all u, v, either $U_1/V_1 = U_2/V_2 = U_3/V_3$ or

$$\frac{(u+v)V_1+i(v-u)V_2+(uv-1)V_3}{(u+v)U_1+i(v-u)U_2+(uv-1)U_3} = \pm \frac{\sqrt{\Sigma V_1^2}}{\sqrt{\Sigma U_1^2}}. \tag{4}$$

where we may assume without restriction that the radicals in (4) are conjugate for conjugate u, v. If the U's are proportional to the V's all six of these functions are constants so that the surface (1) is a point; this condition is also included in (4) so that the latter is the necessary and sufficient condition for the required coincidence.

Choosing first the upper sign in (4) we rewrite it in the form

$$\frac{U_1+iU_2+uU_3}{\sqrt{\Sigma U_1^2}}v+\frac{uU_1-iuU_2-U_3}{\sqrt{\Sigma U_1^2}}=\frac{V_1-iV_2+vV_3}{\sqrt{\Sigma V_1^2}}u+\frac{vV_1+ivV_2-V_3}{\sqrt{\Sigma V_1^2}}. \tag{4'}$$

Since (4′) is an identity in u, v the first number is linear in u, and as there can be no cancellation in its two terms we have

$$\frac{U_1+iU_2+uU_3}{\sqrt{\Sigma U_1^2}}=au+b, \quad \frac{uU_1-iuU_2-U_3}{\sqrt{\Sigma U_1^2}}=c+du, \tag{5}$$

where a, b, c, d are constants. From (4′) it follows that a and c are real, b and d are conjugate. If we take the lower sign in (4) we are led again to (5), but in this case a and c are pure imaginaries, b and $-d$ are conjugate.

The following discussion consists in the study of (5), to which we add the equations, implied in (1), $\Sigma U_1'^2=0$, and

$$U_1'+iU_2'+uU_3'=0. \tag{6}$$

Before undertaking the solution of these equations for U_1, U_2, U_3 we transform (5), writing

$$\sqrt{\Sigma U_1^2}=\rho, \quad U_1=\rho\lambda, \quad U_2=\rho\mu, \quad U_1=\rho\nu,$$

where, evidently,

$$\Sigma\lambda^2=\lambda^2+\mu^2+\nu^2=1, \quad \Sigma\lambda\lambda'=0.$$

From $\Sigma U_1'^2=0$,

$$\rho^2\Sigma\lambda'^2+\rho'^2=0, \quad \rho=e^{\pm i\int\sqrt{\Sigma\lambda'^2}du}.$$

Equations (5) may now be written

$$\lambda+i\mu=u(a-\nu)+b, \quad \lambda-i\mu=\frac{1}{u}(c+\nu)+d. \tag{5'}$$

Our problem may now be regarded as consisting of the solution of (5′) with $\Sigma\lambda^2=1$ for λ, μ, ν, then the determination of ρ and hence U_1, U_2, U_3, to be followed by applying (6) as a necessary condition.

We multiply together equations (5′) and replace $\Sigma\lambda^2$ by unity, finding

$$\nu=\frac{1-ac-bd-adu-bc/u}{a-c-du+b/u}.$$

Differentiating (5′) and multiplying together the resulting equations,

$$\Sigma\lambda'^2=\nu'(a+c)/u-[ac+\nu(a-c)-\nu^2]/u^2.$$

§ 5. **Solution for $b=d=0$, $a\neq c$.** For B_m we find $b=d=0$; it is therefore natural to attempt first the solution of the equations of § 4 on the assumption, $b=d=0$. We suppose further $a\neq c$. We find

$$\nu=\frac{1-ac}{a-c}, \quad \Sigma\lambda'^2=\frac{(1-a^2)(1-c^2)}{u^2(a-c)^2},$$

from which

$$\rho=Au^{\pm i\frac{\sqrt{(1-a^2)(1-c^2)}}{a-c}},$$

where A is an arbitrary constant. The values of λ and μ are found from (5′), and from them and the given values of ν and ρ we obtain U_1, U_2, U_3. We apply to the latter the condition (6); it appears that $a^2=1$ or $a^2c^2=1$. If a^2 or $ac=1$ the surface (1) is plane. It remains to consider only $ac=-1$. Replacing c by $-1/a$ we have two sets of values for the U's corresponding to the double sign in ρ; one of these satisfies (6) only if $a^2=1$, and may be discarded; the other satisfies (6) identically, and is

$$U_1=-\frac{Aa}{2}\,\frac{1-a^2}{1+a^2}\left[u^{\frac{2}{1+a^2}}+\frac{1}{a^2}u^{-\frac{2a^2}{1+a^2}}\right],$$

$$U_2=\frac{Aai}{2}\,\frac{1-a^2}{1+a^2}\left[u^{\frac{2}{1+a^2}}-\frac{1}{a^2}u^{-\frac{2a^2}{1+a^2}}\right],$$

$$U_3=\frac{2Aa}{1+a^2}u^{\frac{1-a^2}{1+a^2}}.$$

These give the equations for B_m if we set

$$a=\pm i\sqrt{\frac{m-1}{m+1}}, \quad A=\mp\frac{ci}{m\sqrt{m^2-1}}.$$

The first of the last two equations gives a real finite value for m unless $a=\pm i$, but such values of a are inadmissible, for both give, from $ac=-1$, $a=c$, a case excluded from our present consideration. The absolute value of m is greater or less than one as a is pure imaginary or real, so that these two cases of B_m, which show many geometrical differences, are also distinguished by the nature of a. It is interesting to observe that $m=0, 1$ are given by $a^2=0, 1$, respectively. These are the values of m excluded in § 3; the value $a^2=0$ is now excluded since $ac=-1$; $a^2=1$ gives for (1) a plane.

§ 6. **Solution for $b=d=0$, $a=c$.** We prove that when $b=d=0$ and $a=c$ the surface (1) is plane. Replacing c by a and writing $b=d=0$ in (5′) we have

$$\lambda+i\mu=(a-\nu)u, \quad \lambda-i\mu=(a+\nu)/u,$$

from which $\Sigma\lambda^2=a^2=1$. We may suppose $a=1$, for changing the sign of a merely changes the signs of λ, μ, ν and hence the signs of x, y, z in (1). It is not now possible to determine ν as in § 4. We find

$$\Sigma\lambda'^2=(\nu^2+2u\nu'-1)/u^2,$$

and substitute this value in the expression for ρ of § 4. Further,

$$2U_1=[1/u+u+\nu(1/u-u)]\rho, \quad 2U_2=[1/u-u+\nu(1/u+u)]\rho i, \quad U_3=\nu\rho;$$

substituting these and the value of ρ in (6),

$$\nu-1=\pm i\sqrt{\nu^2+2u\nu'-1},$$

from which $\nu=cu/(cu-1)$ and $\rho=c'(cu-1)/cu$, where c and c' are constants. Then $U_3=c'$ and (1) is the plane, $z=$constant.

§ 7. **Reduction of the General Case to $b=d=0$.** We now consider (5′) with no hypothesis concerning a, b, c, d. There are two cases: (I) a and c real, b and d conjugate, (II) a and c pure imaginary, b and $-d$ conjugate.

Since equations (5′) involve only u we may regard their transformation as that of the minimal curve,

$$x=U_1, \quad y=U_2, \quad z=U_3.$$

We prove in this section that this curve may be obtained by solving the equations of the preceding section after making a suitable linear transformation of u and a certain rotation of the coordinate axes to which the curve is referred. In the following section we show that the rotation is real, thereby proving that the surface found is unchanged by the transformation.

For any point of the curve the value of ρ is unaffected by a change of variable or by a rotation of the axes. Supposing the new equations of the curve to be

$$x_1=\rho\lambda_1, \quad y_1=\rho\mu_1, \quad z_1=\rho\nu_1,$$

we have $\lambda_1=\alpha_1\lambda+\beta_1\mu+\gamma_1\nu$, with similar equations for μ_1 and ν_1, where the coefficients are the terms of an orthogonal substitution. We write

$$u=\frac{\alpha u_1+\beta}{\gamma u_1+\delta},$$

and prove that by a suitable choice of this substitution and of the rotation (5′) becomes

$$\lambda_1+i\mu_1+u_1\nu_1=a_1u_1, \quad u_1(\lambda_1-i\mu_1)-\nu_1=c_1, \tag{7}$$

where a_1 and c_1 are both real or both pure imaginary, and $a_1c_1=-1$. Substituting for u in (5′) and clearing of fractions,

$$\begin{aligned}
&u_1[\gamma(\lambda+i\mu)+\alpha\nu]+\delta(\lambda+i\mu)+\beta\nu=u_1(a\alpha+b\gamma)+a\beta+b\delta,\\
&u_1[\alpha(\lambda-i\mu)-\gamma\nu]+\beta(\lambda-i\mu)-\delta\nu=u_1(c\gamma+d\alpha)+c\delta+d\beta.
\end{aligned}$$

We make two combinations of the last equations, for the first multiplying the first equation by l, the second by l', and adding; for the second using m and m' as multipliers. We require that the first members of the equations so formed be respectively the first numbers of (7). This condition gives us values for λ_1, μ_1, and two values for ν_1, linear in λ, μ, ν; equating like coefficients in the two values of ν_1,

$$l=-m\delta/\gamma, \quad l'=m\beta/\gamma, \quad m'=-m\alpha/\gamma.$$

Substituting these values in λ_1 and μ_1, then using $\Sigma\lambda_1^2=\Sigma\lambda^2=1$, we find

$$\gamma^2/m^2=(\alpha\delta-\beta\gamma)^2,$$

and choose $m=-\gamma/(\alpha\delta-\beta\gamma)$, and both pairs of multipliers are determined. A rotation has now been found so that for an arbitrary linear substitution for u the first members of the transformed (5′) have the desired form; the coefficients of the substitution may now be chosen to simplify the second members of the equations. Equations (5′) are now replaced by

$$\lambda_1+i\mu_1+u_1\nu_1=a_1u_1+b_1, \quad u_1(\lambda_1-i\mu_1)-\nu_1=c_1+d_1u_1,$$

where

$$\begin{aligned}
a_1\Delta&=a\alpha\delta+b\gamma\delta-c\beta\gamma-d\alpha\beta,\\
b_1\Delta&=a\beta\delta+b\delta^2-c\beta\delta-d\beta^2,\\
c_1\Delta&=-a\beta\gamma-b\gamma\delta+c\alpha\delta+d\alpha\beta,\\
d_1\Delta&=-a\alpha\gamma-b\gamma^2+c\alpha\gamma+d\alpha^2,\\
\Delta&=\alpha\delta-\beta\gamma.
\end{aligned}$$

From these values it appears that

$$a_1+c_1=a+c, \quad a_1c_1-b_1d_1=ac-bd.$$

We wish so to determine $\alpha, \beta, \gamma, \delta$ that $b_1=d_1=0$. This requires, since we must have $\Delta \neq 0$, that β/δ and α/γ be different roots of the quadratic, $dx^2+(c-a)x-b=0$. That the roots of this equation are distinct follows immediately since the discriminant can not vanish when (I) a and c are real, b and d conjugate, or (II) a and c are pure imaginary, b and $-d$ conjugate. Calling the roots of the quadratic x_1, x_2, writing $\alpha=\gamma x_1$, supposing $\delta=1$, as we may if $d\neq 0$, we have

$$a_1=\frac{b+ax_1-cx_2-dx_1x_2}{x_1-x_2}, \quad c_1=\frac{-b-ax_2+cx_1+dx_1x_2}{x_1-x_2}, \quad u=\frac{\gamma x_1u_1+x_2}{\gamma u_1+1}.$$

The values of a_1 and c_1 are independent of γ, so that the surface satisfying the requirement of our problem is applicable to itself in an infinite number of ways. It will appear in the next section that the surface is unchanged by the transformation when γ depends on an arbitrary real parameter. It will also appear that the new Z-axis is independent of γ.

If we apply (6) to the general equations (5') we find, after rather tedious algebraic work, that for all surfaces other than the plane this condition becomes $ac-bd=-1$. This condition might certainly have been foreseen as necessary, for we have observed that $ac-bd$ is invariant under the transformation, and $ac=-1$ if $b=d=0$.

The values of a_1 and c_1 may also be found from the equations,

$$a_1+c_1=a+b, \quad a_1c_1=-1.$$

These give two pairs of values which exchange a_1 and c_1, and correspond to the interchange of x_1 and x_2. These two solutions give to m in B_m values differing only in sign, and therefore lead to the same surface. It may easily be proved that a_1 and c_1 are both real or both pure imaginary with a and c.

§ 8. **Reality of the Rotation.** Evidently to bring the minimal curve,

$$x=V_1, \quad y=V_2, \quad z=V_3,$$

into a corresponding reduced form, v must be transformed by a substitution conjugate to that applied to u, and the coordinate axes subjected to the rotation conjugate to that employed in § 7. To prove that the surface (1) is not changed by these two rotations we show that they are real and therefore identical. This proof may be given in two ways, both of which we indicate. The coefficients of λ, μ, ν in the expressions for λ_1, μ_1, ν_1, mentioned in § 7,

may be expressed in terms of x_1, x_2, γ. The coefficients of ν_1 do not contain γ, so that the new Z-axis is the same for all reducing transformations. It may be shown directly that if (I) a and c are real, b and d conjugate, or (II) a and c are pure imaginary, b and $-d$ conjugate, and if further $\gamma = \sqrt{x_2/x_1}\, e^{i\phi}$, where ϕ is an arbitrary real number, all nine coefficients are real, hence that the surface satisfying the requirement of our problem is a surface B_m, $|m| \neq 1$. It appears also that the coordinate axes to which B_m in its standard form is referred, depend on an arbitrary real parameter, agreeing with a property of the associate surfaces mentioned in § 3. A second method of proving the rotation real consists in showing the rotation identical with its conjugate for the value of γ given above. This may easily be done if we note that $-1/x_2$, $-1/x_1$, $\sqrt{x_2/x_1}\, e^{-i\phi}$ are conjugate respectively to x_1, x_2, $\sqrt{x_2/x_1}\, e^{i\phi}$.

It is interesting to note that the linear transformation connecting u and u_1 is a real rotation of the sphere of the complex variable u for the value of γ given, and also that the transformation connecting the spheres on which u, v and u_1, v_1 are respectively the parameters of the minimal lines is a real rotation.

YALE UNIVERSITY, 1917.

On Integral Invariants.

By F. W. Reed.

The work of Poincaré* on integral invariants has been extended along the same lines by De Donder† and Goursat.‡ Lie§ has written upon a class of linear integral invariants somewhat more general than those considered by Poincaré.

In the present paper the method of infinitesimal transformations developed by Lie‖ is applied to those types defined and discussed by Poincaré. The conditions for invariants of order p are found as a system of ordinary linear differential equations, which are the same as those found by Goursat. The relation between the solutions of the systems of various orders is brought out in full. In particular it is shown how all the linear invariants depend upon the original system and how all the invariants of higher order can be constructed when these are known. An extension of Poisson's theorem is given. Further it is shown that the least action integral of dynamics is nothing other than an integral invariant of the equations of motion.

§1. *Invariants of Order p.*

Consider the multiple integral

$$I_p = \int^p \Sigma A_{\alpha_1 \ldots \alpha_p} dx_{\alpha_1} dx_{\alpha_2} \ldots dx_{\alpha_p}, \tag{a}$$

where

$$A_{\alpha_1 \ldots \alpha_p} = \pm A_{\alpha'_1 \ldots \alpha'_p},$$

the upper or lower sign being taken according as the rearrangement of the $\alpha_1 \ldots \alpha_p$ in the new order $\alpha'_1 \ldots \alpha'_p$ is obtained by an even or odd number of consecutive changes, and the Σ sign is extended to all the combinations of the

* Poincaré, *Journal de l'École Polytechnique*, 2e série, premier cahier (1895). "Les Méthodes Nouvelles de la Mécanique Céleste, Tome III; *Acta Mathematica*, 13 (1890).

† De Donder, *Rendiconti del Circolo Matematico di Palermo*, 15, 16 (1901, 1902).

‡ Goursat, *Journal de Mathématiques pures et appliquées*, 6e série, Tome IV (1908).

§ Lie, "Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 49 (1897), pp. 342, 369.

‖ Lie, "Theorie der Transformationsgruppen."

n letters $\alpha_1 \ldots \alpha_n$ taken p at a time. This integral is equivalent to the integral

$$I_p = \int^p \Sigma A_{\alpha_1 \ldots \alpha_p} \frac{\partial(x_{\alpha_1} \ldots x_{\alpha_p})}{\partial(y_1 \ldots y_p)} dy_1 \ldots dy_p. \tag{b}$$

The interchange of x_{α_i} and $x_{\alpha_{i+1}}$ in any one of the functional determinants changes the sign of the corresponding term in (b), but the sign may be restored by making the same permutation of α_i and α_{i+1} in the subscript of $A_{\alpha_1 \ldots \alpha_p}$. The same changes have the same effect upon any particular term of (a). It is sufficient, therefore, when a choice of the subscripts of $A_{\alpha_1 \ldots \alpha_p}$ is once made to write the subscripts in $dx_{\alpha_1} \ldots dx_{\alpha_p}$ in the *same order*.

To facilitate the writing the following symbols are introduced:

$$(1 \ldots p-2, \lambda, p) \equiv A_{\alpha_1 \ldots \alpha_{p-2}, \lambda, \alpha_p}, \text{ etc.}$$

$$[1, \lambda] = \frac{\partial X_{\alpha_1}}{\partial x_\lambda}, \text{etc.} \quad [\lambda, p] = \frac{\partial X_\lambda}{\partial x_{\alpha_p}}.$$

Under the transformation

$$dx_i = X_i dt, \tag{1}$$

or

$$X(f) = \frac{\partial f}{\partial x_1} X_1 + \frac{\partial f}{\partial x_2} X_2 + \ldots + \frac{\partial f}{\partial x_n} X_n,$$

the integral (a) becomes

$$I'_p = \int^p \sum_{\alpha_1 \ldots \alpha_p} [(1 \ldots p) + X(1 \ldots p) dt] [dx_{\alpha_1} + \Sigma [1, \lambda] dx_\lambda dt]$$
$$[dx_{\alpha_2} + \Sigma [2, \lambda] dx_\lambda dt] \ldots [dx_{\alpha_p} + \Sigma [p, \lambda] dx_\lambda dt]. \tag{a'}$$

The term of zero degree in dt is I_p of (a). Equating to zero the coefficient of $dx_{\alpha_1} \ldots dx_{\alpha_p}$ in the term of first degree in dt we have

$$\Sigma [X(1 \ldots p) + (1 \ldots p-1, \lambda) [\lambda, p] + (1 \ldots p-2, \lambda, p) [\lambda, p-1]$$
$$+ \ldots + (\lambda, 2 \ldots p) [\lambda, 1]] = 0. \tag{2}$$

If these conditions are satisfied I_p is an integral invariant under the transformation.

THEOREM I. *If an I_1 and an I_p are known, an I_{p+1} may be written by means of the relation*

$$(1 \ldots p+1) = (1)(2 \ldots p+1) - (2)(1, 3 \ldots p+1) + \ldots \pm (p+1)(1 \ldots p).$$

Substitute this expression for $(1 \ldots p+1)$ in the general condition (2). The respective terms become

$$X(1 \ldots p+1) = X((1)(2 \ldots p+1) - (2)(1, 3 \ldots p+1)$$
$$+ \ldots \pm (p+1)(1 \ldots p)),$$

$$\Sigma(1\ldots p,\lambda)[\lambda,p+1]=\Sigma[(1)(2\ldots p,\lambda)-(2)(1,3\ldots p,\lambda)$$
$$+\ldots\pm(\lambda)(1\ldots p)][\lambda,p+1],\quad \lambda(=)\alpha_{p+1},$$
$$\Sigma(1\ldots p-1,\lambda,p+1)[\lambda,p]=\Sigma[(1)(2\ldots p-1,\lambda,p+1)$$
$$-(2)(1,3\ldots p-1,\lambda,p+1)$$
$$+\ldots\pm(p+1)(1\ldots p-1,\lambda)][\lambda,p],\quad \lambda(=)\alpha_p,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Sigma(1,\lambda,3\ldots p+1)[\lambda,2]=\Sigma[(1)(\lambda,3\ldots p+1)-(\lambda)(1,3\ldots p+1)$$
$$+\ldots\pm(p+1)(1,\lambda,3\ldots p)][\lambda,2],\quad \lambda(=)\alpha_2,$$
$$\Sigma(\lambda,2\ldots p+1)[\lambda,1]=\Sigma[(\lambda)(2\ldots p+1)-(2)(\lambda,3\ldots p+1)$$
$$+\ldots\pm(p+1)(\lambda,2\ldots p)][\lambda,1],\quad \lambda(=)\alpha_1,$$

where $\lambda(=)\alpha_i$ means λ different for all the $\alpha_1\ldots\alpha_{p+1}$ except α_i. By adding to the second equation

$$0=(1)(2\ldots p,1)[1,p+1]-(2)(1,3\ldots p,2)[2,p+1]$$
$$+\ldots\pm(p)(1\ldots p-1,p)[p,p+1]$$
$$\pm[(1)(1\ldots p)[1,p+1]+(2)(1\ldots p)[2,p+1]$$
$$+\ldots+(p)(1\ldots p)[p,p+1]],$$

the gaps are filled and we have

$$\Sigma(1\ldots p,\lambda)[\lambda,p+1]=\Sigma[(1)(2\ldots p,\lambda)-(2)(1,3\ldots p,\lambda)$$
$$+\ldots\pm(\lambda)(1\ldots p)[\lambda,p+1]],\quad \lambda(=)\alpha_{p+1},$$

where λ takes all values consistent with the properties of $(1\ldots p)$ previously indicated. The remaining equations may be reduced in a similar manner. Adding now the first terms in the right members of the equations thus reduced the sum is

$$(1)\Sigma[X(2\ldots p+1)+(2\ldots p,\lambda)[\lambda,p+1]+\ldots+(\lambda,3\ldots p+1)[\lambda,2]]$$
$$+(2\ldots p+1)\Sigma[X(1)+(\lambda)[\lambda,1]].$$

This expression is zero when the (i) and the $(1\ldots p)$ are coefficients in an I_1 and an I_p, respectively. The sum of the second terms vanish in a similar manner, etc. Thus the conditions for an I_{p+1} are fulfilled.

THEOREM II. *The product of p linearly independent invariants I_1 is an I_p.*

Suppose p linear invariants are known

$$I_1^{(i)}=\int L_i\qquad (i=1,2\ldots p),$$

where

$$L_1=A_1^1dx_1+\ldots+A_n^1dx_n=(1)^1dx_1+\ldots+(n)^1dx_n,$$
$$L_2=A_1^2dx_1+\ldots+A_n^2dx_n=(1)^2dx_1+\ldots+(n)^2dx_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$L_p=A_1^pdx_1+\ldots+A_n^pdx_n=(1)^pdx_1+\ldots+(n)^pdx_n.$$

The determinants of the matrix of the right-hand members of the system give all the terms of this product since products in the same row could not enter, and those from the same column give zero on integration. Further, the coefficients of $dx_{\alpha_1}\ldots dx_{\alpha_p}$ is $\Sigma\pm(1)^1(2)^2\ldots(p)^p$, where the summation means all the permutations of $\alpha_1\ldots\alpha_p$, and the upper or lower sign is to be taken according as it requires an even or an odd number of changes to bring an element of this sum to the order $\alpha_1\ldots\alpha_p$. Consequently, remembering the properties of $(1\ldots p)$ previously defined, we have

$$\Sigma\pm(1)^1(2)^2\ldots(p)^p=(1\ldots p).$$

COROLLARY 1. *If the L_i are not linearly independent the resulting I_p is identically zero.*

COROLLARY 2. *Since the determinants may be expanded with respect to their minors of any order we have*

$$(1\ldots p)(p+1\ldots p+q)=(1\ldots p+q)$$

which gives symbolically $I_{p+q}=I_pI_q$.

COROLLARY 3. *When p is odd $(I_p)^2\equiv 0$.*

COROLLARY 4. *The number of coefficients in an I_p is $\dfrac{n!}{p!(n-p)!}$, the same as the number of terms in the product $L_1L_2\ldots L_p$, consequently, when n independent linear invariants are known all the I_p can be written.*

The converse of this last corollary is not true. For example, the invariant of the second order $I_1^1I_1^2+I_1^3I_1^4$ is not factorable.

THEOREM III. *Knowing an I_{p+1}, an I_p can be found by means of the formula*

$$(1\ldots p)=\Sigma(1\ldots p,\sigma)X_\sigma.$$

To verify this formula substitute it in equation (2). We get

$$\begin{aligned}
\Sigma[X(1\ldots p,\sigma)\cdot X_\sigma+(1\ldots p,\sigma)X(X_\sigma),&\quad \sigma\neq\alpha_1\ldots\alpha_p,\\
+\sum_\lambda[\lambda,p]\cdot\sum_\sigma(1\ldots p-1,\lambda,\sigma)X_\sigma,&\quad \sigma\neq\alpha_1\ldots\alpha_{p-1},\lambda,\\
+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,&\\
+\sum_\lambda[\lambda,1]\cdot\sum_\sigma(\lambda,2\ldots p,\sigma)X_\sigma=0],&\quad \sigma\neq\lambda,\alpha_2\ldots\alpha_p.
\end{aligned}$$

The subscripts given by $(1,2,\ldots)$ must be chosen so that there are no repetitions. The coefficient of X_ρ, where $\rho\neq\alpha_1\ldots\alpha_p$ is

$$X(1\ldots p,\rho)+\sum_\sigma(1\ldots p,\sigma)[\sigma,\rho]$$
$$+\sum_\lambda(1\ldots p-1,\lambda,\rho)[\lambda,p]+\ldots+\sum_\lambda(\lambda,2\ldots p,\rho)[\lambda,1]=0.$$

These are the conditions for an I_{p+1}. The remaining terms are zero, for we have as coefficients of X_{α_i}, $(i=1, 2\ldots p)$

$$\sum_{\sigma}(1\ldots p, \sigma)[\sigma, i]+\sum_{\lambda}(1\ldots i-1, \lambda, i+1\ldots p, i)[\lambda, i].$$

Here the coefficient of any $[\sigma, i]$ is identically zero since interchanging λ and α_i is accomplished by $j+(j-1)$ moves. This number is odd.

The application of this theorem to the problem of finding invariants of order $p-1$ from those of order $p+1$ leads to coefficients identically zero. We have

$$(1\ldots p)=\Sigma(1\ldots p, \lambda)X_{\lambda}, \qquad \lambda \neq \alpha_1\ldots\alpha_p,$$

also

$$(1\ldots p-1)=\Sigma(1\ldots p-1, \sigma)X_{\sigma}, \qquad \sigma \neq \alpha_1\ldots\alpha_{p-1}.$$

Substituting the first equation in the second it becomes

$$(1\ldots p-1)=\sum_{\sigma}X_{\sigma}\sum_{\lambda}(1\ldots p-1, \sigma, \lambda)X_{\lambda}, \qquad \lambda \neq \sigma.$$

The coefficient of $X_{\sigma}X_{\lambda}$ is

$$(1\ldots p-1, \sigma, \lambda)+(1\ldots p-1, \lambda, \sigma)\equiv 0.$$

If the equations are canonical then

$$I_2=\int dx_1dx_2+dx_3dx_4+\ldots$$

as can be verified readily. Applying Theorem III, we have

$$A_1=X_2, \quad A_2=-X_1, \ldots,$$

$$I_1=\int -\frac{\partial F}{\partial x_1}dx_1-\frac{\partial F}{\partial x_2}dx_2-\ldots-\frac{\partial F}{\partial x_n}dx_n.$$

The condition for the invariant of the *n-th* order is

$$X(1\ldots n)+(1\ldots n)\left[\frac{\partial X_1}{\partial x_1}+\ldots+\frac{\partial X_n}{\partial x_n}\right]=0,$$

which becomes

$$\Sigma\frac{\partial}{\partial x_i}\{(1\ldots n)X_i\}=0,$$

showing that $(1\ldots n)$ is a multiplier of (1). In particular, if the equations are canonical,

$$\Sigma\frac{\partial X_i}{\partial x_i}=0, \text{ and } (1\ldots n)=\text{const.}=1$$

is a multiplier.

The invariant of order $n-1$ derived from this is identically zero.

§ 2. *Invariants of the Type* $\int\sqrt[p]{\Sigma(1\ldots.p)\,dx_{a_1}\ldots.dx_{a_p}}$.

Let

$$I_p=\int\sqrt[p]{\sum_{a_1\ldots.a_p}(i\ldots.p)\,dx_{a_1}\ldots.dx_{a_p}}=\int\sqrt[p]{A}.$$

We have

$$X(A^{\frac{1}{p}})=\frac{1}{p}\cdot A^{\frac{1}{p}-1}X(A).$$

Then I_p will be an integral invariant when

$$X(A)=0.$$

This condition, when expanded, is identical with the condition (2). However, in the present case, $(1\ldots.p)$ has the same value for all permutations of $\alpha_1\ldots.\alpha_p$, and may be different from zero when $\alpha_i=\alpha_j$ in the subscript. The conditions to be satisfied are therefore

$$\Sigma[X(1\ldots.p)+(1\ldots.p-1,\lambda)[\lambda,p]+\ldots.+(\lambda,2\ldots.p)[\lambda,1]]=0,$$

where λ takes all values 1 to n.

THEOREM IV. *An I_{p+q} can be constructed from an I_p and an I_q by means of the formula*

$$(1\ldots.p+q)=(1\ldots.p)(p+1\ldots.p+q).$$

Substituting this expression in the conditions for an I_{p+q}, namely,

$$X(1\ldots.p+q)+(1\ldots.p+q-1,\lambda)[\lambda,p+q]$$
$$+\ldots.+(\lambda,2\ldots.p+q)[\lambda,1]=0,$$

it becomes

$$(1\ldots.p)X(p+1\ldots.p+q)+(p+1\ldots.p+q)X(1\ldots.p)$$
$$+(1\ldots.p)(p+1\ldots.p+q-1,\lambda)[\lambda,p+q]$$
$$+\ldots.+(1\ldots.p)(\lambda,p+2\ldots.p+q)[\lambda,p+1]$$
$$+(1\ldots.p-1,\lambda)(p+1\ldots.p+q)[\lambda,p]$$
$$+\ldots.+(\lambda,2\ldots.p)(p+1\ldots.p+q)[\lambda,1]=0,$$

and this condition is satisfied since

$$X(1\ldots.p)+(1\ldots.p-1,\lambda)[\lambda,p]+\ldots.+(\lambda,2\ldots.p)[\lambda,1]=0$$

and

$$X(p+1\ldots.p+q)+(p+1\ldots.p+q-1,\lambda)[\lambda,p+q]$$
$$+\ldots.+(\lambda,p+2\ldots.p+q)[\lambda,p+1]=0.$$

COROLLARY. *If*

$$I_1^1=\int L_1,\quad I_1^2=\int L_2,\ \ldots.,\quad I_1^p=\int L_p$$

are p linear integral invariants of (1) *then*

$$\int\sqrt[p]{L_1L_2\ldots.L_p}$$

is an integral invariant.

§ 3. *Relative Invariants.*

The following definitions and theorems are due to Poincaré:

(a). An I_p extended to a closed multiplicity E_p in a space of n dimensions can be replaced by an I_{p+1} extended to a multiplicity E_{p+1} in a space of n dimensions limited by the E_p. The coefficients of the I_{p+1} are found from those of the I_p by means of the relations

$$(1\dots p+1)=\frac{\partial(1\dots p)}{\partial x_{a_{p+1}}}\pm\frac{\partial(2\dots p,p+1)}{\partial x_{a_1}}+\dots\pm\frac{\partial(p+1,1\dots p-1)}{\partial x_{a_p}},$$

the upper or lower signs being taken according as p is even or odd.

(b). The expression under the integral sign of I_p is an exact differential when the $(1\dots p+1)$ are zero.

(c). The I_{p+1} of (a) is an integral of an exact total differential. This is the generalized Stokes' theorem.

Let $A'_{a_1\dots a_p}\equiv(1\dots p)'$ represent the left-hand member of equation (2). The increment of I_p was found to be

$$[\textstyle\int^p\sum\limits_{a_1\dots a_p}(1\dots p)'dx_{a_1}\dots dx_{a_p}]\,dt.$$

If

$$\Sigma(1\dots p)'dx_{a_1}\dots dx_{a_p}$$

is exact, and if its integral be extended to a closed E_p, the resulting equivalent I_{p+1} is identically zero. In this case the original I_p is a relative integral invariant of (1). We shall write it J_p. When

$$(1\dots p)'=0, \tag{2}$$

we have an absolute invariant I_p; when

$$\Sigma(1\dots p)'dx_{a_1}\dots dx_{a_p}\ \textit{is exact}, \tag{6}$$

we have a relative invariant J_p. By means of the equation (5) the statement (6) may be written in full as an equation,

$$\frac{\partial(1\dots p)'}{\partial x_{a_{p+1}}}\pm\frac{\partial(2\dots p,p+1)}{\partial x_{a_1}}+\dots\pm\frac{\partial(p+1,1\dots p-1)}{\partial x_{a_p}}=0. \tag{7}$$

The relation between these two types of invariants may be illustrated by some simple examples.

EXAMPLE 1. Let equations (1) have the canonical form

$$\frac{dx_1}{dt}=\frac{\partial F}{\partial x_2},\quad \frac{dx_2}{dt}=-\frac{\partial F}{\partial x_1},\quad \frac{dx_3}{dt}=\frac{\partial F}{\partial x_4},\quad \frac{dx_4}{dt}=-\frac{\partial F}{\partial x_3}.$$

We have

$$A'_i=\Sigma\left[X(A_i)+A_k\frac{\partial X_k}{\partial x_i}\right]\qquad(i,\,k=1,\,2,\,3,\,4).$$

The integral

$$\int x_2 dx_1 + x_4 dx_3$$

is a relative invariant, since

$$\frac{\partial A'_i}{\partial x_k} - \frac{\partial A'_k}{\partial x_i} = 0.$$

It is not, however, an I_1, since $A'_i \neq 0$.

EXAMPLE 2. Let equations (1) have the canonical form

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i} = X_i, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} = Y_i \qquad (i=1, 2, \ldots, n).$$

Where F is homogeneous of degree 1 in y_i, or

$$F = \Sigma y_i \frac{\partial F}{\partial y_i} = \Sigma y_i X_i.$$

The partial derivatives of this equation with respect to x_i, y_i are

$$Y_i + \sum_k y_k \frac{\partial X_k}{\partial x_i} = 0, \quad \sum_k y_k \frac{\partial X_i}{\partial y_k} = 0,$$

and these are the necessary and sufficient conditions that

$$I = \int \Sigma y_i dx_i$$

is an absolute invariant.

According to Goursat we designate as D the operation of passing from an I_p to an I_{p+1} by means of equation (5), and as E the operation of passing from an I_p to an I_{p-1} by means of equation (4).

THEOREM V. *The operations D and E are in general not inverse.*

Consider the case of a linear invariant, $I_1 = \int a x_1 + b x_2$ of the linear system of equations in x_1, x_2. By identifying (5) with (4) we have

$$a = \left(\frac{\partial a}{\partial x_2} - \frac{\partial b}{\partial x_1}\right) X_2, \quad b = -\left(\frac{\partial a}{\partial x_2} - \frac{\partial b}{\partial x_1}\right) X_1, \tag{8}$$

where a, b must satisfy the conditions (2); but (2) are satisfied only by $a = \frac{\partial F}{\partial x_1}$, $b = \frac{\partial F}{\partial x_2}$ where F is an integral of the system. With this substitution, however, the second members of (8) become identically zero, consequently, an I_1 satisfying the conditions (8) and (2) does not exist.

§ 4. *Extension of Poisson's Theorem.*

If equations (1) are in canonical form

$$\frac{dx_i}{dt}=\frac{\partial F}{\partial y_i},\quad \frac{dy_i}{dt}=-\frac{\partial F}{\partial x_i}\qquad (i=1, 2, \ldots, n), \tag{1}$$

where F does not contain t explicitly, then $F=$ const. is an integral of (1). Further, if we put

$$[F_1, F_2]\equiv\Sigma\left(\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial y_i}-\frac{\partial F_1}{\partial y_i}\frac{\partial F_2}{\partial x_i}\right),$$

we have

$$\frac{dF_1}{dt}=[F_1, F]$$

by means of (1). The necessary and sufficient condition that $F_1=$ const. be an integral of (1) is that $[F_1, F]=0$. Let

$$I_1=\int\Sigma(M_i dx_i+N_i dy_i)$$

be a linear integral invariant of (1), then the conditions to be satisfied are

$$\frac{dM_k}{dt}+\sum_i\left(M_i\frac{\partial^2 F}{\partial y_i\partial x_k}-N_i\frac{\partial^2 F}{\partial x_i\partial x_k}\right)=0, \tag{m}$$

$$\frac{dN_k}{dt}+\sum_i\left(M_i\frac{\partial^2 F}{\partial y_i\partial y_k}-N_i\frac{\partial^2 F}{\partial x_i\partial y_k}\right)=0. \tag{n}$$

The coefficients of another similar invariant may be represented by M'_i, N'_i determined by the same conditions and lettered (m′) (n′). These $4n$ equations when multiplied by

$$N'_1, \ldots N'_n,\ -M'_1, \ldots -M'_n,\ -N_1, \ldots -N_n,\ M_1, \ldots M_n,$$

respectively, give on adding,

$$\frac{d}{dt}\Sigma(M_iN'_i-N_iM'_i)=0.$$

Therefore,

$$\Sigma(M_iN'_i-N_iM'_i)=\text{const.}$$

Remembering that

$$M_i=\frac{\partial F_1}{\partial x_i},\quad M'_i=\frac{\partial F_2}{\partial x_i},$$

$$N_i=-\frac{\partial F_1}{\partial y_i},\quad N'_2=-\frac{\partial F_2}{\partial y_i},$$

we have

$$\Sigma\left(\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial y_i}-\frac{\partial F_1}{\partial y_i}\frac{\partial F_2}{\partial x_i}\right)\equiv[F_1, F_2]=\text{const}$$

It is well known (theorem of Poisson) that $[F_1, F_2]$ is an integral of (1) when F_1, F_2 are integrals. The above result can be expressed in determinant form

$$\Sigma \begin{vmatrix} M_i & M'_i \\ N_i & N'_i \end{vmatrix} = \text{const.}$$

Consider the case of four known integrals of (1). Then we have

$$\begin{vmatrix} M_i^1 & M_i^2 & M_i^3 & M_i^4 \\ N_i^1 & N_i^2 & N_i^3 & N_i^4 \\ M_k^1 & M_k^2 & M_k^3 & M_k^4 \\ N_k^1 & N_k^2 & N_k^3 & N_k^4 \end{vmatrix} = \text{const.}$$

since all the minors of the second order found in the first and second rows, also in the third and fourth rows, are constant.

And generally

$$\sum_{\alpha_1}\sum_{\alpha_2}\cdots\sum_{\alpha_p} \begin{vmatrix} M_{\alpha_1}^1 & M_{\alpha_1}^2 \dots & M_{\alpha_1}^{2p} \\ N_{\alpha_1}^1 & N_{\alpha_1}^2 \dots & N_{\alpha_1}^{2p} \\ \vdots & & \\ M_{\alpha_p}^1 & M_{\alpha_p}^2 \dots & M_{\alpha_p}^{2p} \\ N_{\alpha_p}^1 & N_{\alpha_p}^2 \dots & N_{\alpha_p}^{2p} \end{vmatrix} \equiv \sum_{\alpha_1 \dots \alpha_p} \Delta_{\alpha_1 \dots \alpha_p} = \text{const.}$$

We may assume all the α_i different in making the summation, because otherwise the particular determinant involved would be zero.

In particular if $p = n$ we have, since the total number of determinants in the sum is $n!$, and, since the permutation of the numbers $\alpha_1 \dots \alpha_p$ corresponds to a transposition of the rows keeping rows $2k-1$, $2k$ together,

$$\sum_{\alpha_1 \dots \alpha_n} \Delta_{\alpha_1 \dots \alpha_n} = n!\,\Delta_{\alpha_1 \dots \alpha_n} = n!\,\Delta_{1,2 \dots n}.$$

$\Delta_{1 \dots n}$ is simply the functional determinant of the F_i with respect to the x_i, y_i. This is the extension of Poisson's theorem.

§ 5. *Least Action.*

Consider the canonical equations of dynamics

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

which admit the relative invariant

$$J_1 = \int \Sigma p_i dq_i,$$

and the equivalent absolute invariant (by Stokes' theorem)

$$I_2 = \int\int \Sigma dp_i dq_i.$$

Goursat[*] has shown that the integral invariant of *p-th* order

$$I_p=\int_{E_p}\Sigma A_{a_1\ldots\ldots a_p}dx_{a_1}\ldots\ldots dx_{a_p}$$

of the system

$$\frac{dx_i}{dt}=X_i \qquad (i=1,\,2,\,\ldots\ldots,\,n)$$

is identical with

$$I_p=\int_0^T dt\int_{E_{p-1}}\Sigma C_{a_1\ldots\ldots a_{p-1}}dx_{a_1}\ldots\ldots dx_{a_{p-1}},$$

where

$$C_{a_1\ldots\ldots a_{p-1}}=\sum_{i=1}^{n}A_{a_1\ldots\ldots a_{p-1},\,i}X_i.$$

By specializing this result so as to apply to the above canonical system and its I_2 we have

$$I_2=\int dt\int -dH.$$

Now, if this integral be extended to any E_2 for which $H=$ const., we have

$$I_2=0,\quad J_1=0.$$

It is to be noted that in this transformation of I_2 the form has been simplified *by means of the differential equations.* We can now prove the following:

THEOREM VI. *The relative invariant J_1 represents twice the action in a dynamical system, that is, $\int Tdt$.*

We have $H=T-U$ where T is independent of the q_i and homogeneous of the second degree in the p_i, and U is a function of the q_i only. Then we have

$$J_1=\int\Sigma p_i dq_i=\int\Sigma p_i\frac{\partial H}{\partial p_i}\,dt=\int 2Tdt.$$

UNIVERSITY OF ILLINOIS, *September*, 1916.

---

[*] Goursat, *Journal de Mathématiques pures et appliquées*, 7 série, Tome I (1915).

Fundamental Regions for Certain Finite Groups in S_4.

BY HENRY F. PRICE.

One of the most interesting results of the study of transformations is what Klein has termed the "fundamental region."

A fundamental region for a group of transformations is a system of points which contains one and only one point of every conjugate set.

Fundamental regions in the complex plane have been studied for some time and are well known. Klein * and his followers have developed the subject to a considerable extent. There is a close relationship between this subject and the elliptic modular functions and the reduction of quadratic forms.

The fundamental regions for groups in more than one complex variable have not been studied much. However, J. W. Young,† in a recent paper, obtained such regions for cyclic groups in two complex variables.

In this paper will be considered fundamental regions for certain finite groups in two complex variables. The octahedral and icosahedral groups will be dealt with.

The fundamental regions for these groups in the real plane can be readily determined and found to be triangles bounded by the axes of reflections. In the case of the complex plane the problem is solved by using Hermitian forms which meet the real plane in the sides of these triangles.

The problem will be solved completely in the case of the octahedral group. In the case of the icosahedral group it will be solved except for the points which reduce one or more of the Hermitian forms to zero.

The ternary collineation group G_{24} can be generated by the following three operations: $E_1[-\xi_1, \xi_3, \xi_2]$, $E_2[\xi_2, \xi_1, \xi_3]$ and $E_3[\xi_1, \xi_3, \xi_2]$.

It permutes the points of the real plane. As it contains nine operations of order 2, there are nine reflections. As the group is simply isomorphic with the symmetric group on four letters, it is evident that these reflections are in

* Felix Klein, "Elliptischen Modulfunktionen, Vol. I, pp. 183–207.

† J. W. Young, "Fundamental Regions for Cyclical Groups of Linear Fractional Transformations on Two Complex Variables," *Bull. Amer. Math. Soc.*, Vol. XVII, p. 340.

two conjugate sets. The axes of the reflections, $\xi_1 = \pm\xi_3$, $\xi_2 = \pm\xi_3$, $\xi_1 = \pm\xi_2$, $\xi_1 = 0$, $\xi_2 = 0$ and $\xi_3 = 0$, divide the plane into twenty-four triangles.

Any point in one of these triangles can be transformed, by a suitably chosen operation of the group, into a point in any other triangle. Any triangle is then a *fundamental region* in the plane for the group G_{24}.

The group permutes the complex points of the plane also. If we consider the totality of points in the plane, complex as well as real, as real points in four-space, we may ask the question whether fundamental regions exist in S_4 for the group under consideration. If one does exist it must contain one triangle, and only one, of the real plane. The fixed points of the transformations would lie on the boundaries of such a fundamental region.

Consider the Hermitian forms $\xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2$ and $\xi_1\bar{\xi}_2 + \bar{\xi}_1\xi_2$ in which $\dfrac{\xi_1}{\xi_3} = x + iu$ and $\dfrac{\xi_2}{\xi_3} = y + iv$. Under the G_{24} we have two conjugate sets of three forms each:

$$\begin{array}{lll} (1)\ \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2, & & (4)\ \xi_1\bar{\xi}_2 + \bar{\xi}_1\xi_2, \\ (2)\ \xi_2\bar{\xi}_2 - \xi_3\bar{\xi}_3, & \text{and} & (5)\ \xi_2\bar{\xi}_3 + \bar{\xi}_2\xi_3, \\ (3)\ \xi_3\bar{\xi}_3 - \xi_1\bar{\xi}_1, & & (6)\ \xi_3\bar{\xi}_1 + \bar{\xi}_3\xi_1. \end{array}$$

It is evident that there is at least one relation between the forms, *i. e.*, $(1) + (2) + (3) \equiv 0$.

If we consider the portion of S_4 in which the signs of (1), (3), (4), (5) and (6) are all $+$ and make use of the relation $(1) + (2) + (3) \equiv 0$, we see that the sign of (2) is determined as $-$.

This region will be written $[+-+, +++]$ where the signs of the six forms are written in order.

We shall next consider into how many such regions S_4 is divided by the six Hermitian forms.

The forms (1), (2) and (3) are conjugate under G_{24}, and because of the relation $(1) + (2) + (3) \equiv 0$ admit at most six arrangements of sign. The forms (4), (5) and (6) are also conjugate under G_{24}, and admit at most eight arrangements of sign. There are therefore forty-eight possible choices of sign for (1), (2) .... (6). But the eight arrangements divide into two complete conjugate sets of four under G_{24} according as the number of $+$ signs is odd or even. For one of such four choices for the forms (4), (5), (6), the six arrangements of sign for (1), (2), (3) all are conjugate, *e. g.*, the $+++$ choice is unaltered by the group G_6 of permutations of the variables, and the G_6 permutes all arrangements for (1), (2), (3).

Hence the forty-eight possible choices of sign or possible regions in S_4 divide into two conjugate sets of twenty-four each, and two of these regions, one from each set, constitute a fundamental region in S_4 for G_{24}.

The region $[+-+,\ +++]$ belongs to one of the sets. By changing the sign of (5) to $-$ we obtain a region $[+-+,\ +-+]$ of the other set.

Taking these two regions together we obtain $\Gamma=[+-+,\ +\pm+]$ which is a fundamental region in S_4 for the group G_{24}, except for the points which reduce one or more of the Hermitian forms to zero.

If we consider the points which reduce one or more of the forms to zero we are dealing with what we may call the "boundaries" of the fundamental regions.

By placing the six forms in Γ equal to zero singly, in pairs, in groups of three, etc., in all possible ways, and discarding those which are conjugates of others, it is found that there are twenty-three sets of points which are sections of Γ's boundaries and which should be taken in the fundamental region for the group. Γ can be defined completely, therefore, by the sets of points:

$$[+-+,\ +\pm+],$$
$$\begin{array}{llll}
[+-+,\ +0+], & [0-+,\ +\pm+], & [+-0,\ +\pm+], & [+-+,\ 0++], \\
[+-+,\ ++0], & [0-+,\ +0+], & [+-0,\ +0+], & [+-+,\ 000], \\
[+-+,\ 0+0], & [0-+,\ 0++], & [+-0,\ ++0], & [000,\ +\pm+], \\
[+-+,\ 00+], & [0-+,\ 00+], & [+-0,\ 00+], & [000,\ +0+], \\
[+-+,\ +00], & [0-+,\ +00], & [+-0,\ 0+0], & [000,\ 00+], \\
 & [0-+,\ 000], & [+-0,\ 000]. &
\end{array}$$

The ternary collineation group G_{60} furnishes a more complex fundamental region in S_4 than G_{24} does. It is well known that this group can be generated by the three operations:

$$E_1[\xi_2,\ \xi_3,\ \xi_1];\quad E_2[\xi_1,\ -\xi_2,\ -\xi_3];$$

and

$$E_3\begin{cases}\xi_1'=\ \ \xi_1-\alpha\xi_2+(\alpha+1)\xi_3, \\ \xi_2'=-\alpha\xi_1+(\alpha+1)\xi_2+\xi_3, \\ \xi_3'=\ \ (\alpha+1)\xi_1+\xi_2-\alpha\xi_3,\end{cases}$$

where $\alpha=\dfrac{-1\pm\sqrt{5}}{2}$.*

This group permutes the points of the real plane. As it contains fifteen operations of order 2, there are fifteen axes of reflections. These lines divide

* H. H. Mitchell, "Determination of the Ordinary and Modular Ternary Linear Groups," *Trans. Amer. Math. Soc.*, Vol. XII, No. 2, p. 223.

the plane into sixty triangles. The intersections of these fixed lines are *real* fixed points of three classes; first, the points left invariant under the fifteen subgroups of order 4; second, the points invariant under the ten subgroups of order 6; and third, the points invariant under the six subgroups of order 10.

Each of the sixty triangles into which the plane is divided by the fifteen fixed lines has for its vertices one of each of the three classes of fixed points.

Each of the sixty triangles is a fundamental region in the plane. The group also permutes the complex points of the plane. Just as in the case of the G_{24} we can consider the totality of real and complex points in the plane as real points in S_4 and seek a fundamental region for the group G_{60} in the higher space.

Consider the Hermitian form $2\xi_1\bar{\xi}_2+2\bar{\xi}_1\xi_2$. Under G_{60} there is a single set of fifteen forms conjugate to $2\xi_1\bar{\xi}_2+2\bar{\xi}_1\xi_2$, which can be expressed in terms of six forms:

$$\begin{aligned}
F_1 &= \quad(\alpha-1)\xi_1\bar{\xi}_1+(\alpha+2)\xi_2\bar{\xi}_2-(2\alpha+1)\xi_3\bar{\xi}_3+3\xi_1\bar{\xi}_2+3\bar{\xi}_1\xi_2,\\
F_2 &= \quad(\alpha+2)\xi_1\bar{\xi}_1-(2\alpha+1)\xi_2\bar{\xi}_2+(\alpha-1)\xi_3\bar{\xi}_3-3\xi_3\bar{\xi}_1-3\bar{\xi}_3\xi_1,\\
F_3 &= -(2\alpha+1)\xi_1\bar{\xi}_1+(\alpha-1)\xi_2\bar{\xi}_2+(\alpha+2)\xi_3\bar{\xi}_3+3\xi_2\bar{\xi}_3+3\bar{\xi}_2\xi_3,\\
F_4 &= -(2\alpha+1)\xi_1\bar{\xi}_1+(\alpha-1)\xi_2\bar{\xi}_2+(\alpha+2)\xi_3\bar{\xi}_3-3\xi_2\bar{\xi}_3-3\bar{\xi}_2\xi_3,\\
F_5 &= \quad(\alpha+2)\xi_1\bar{\xi}_1-(2\alpha+1)\xi_2\bar{\xi}_2+(\alpha-1)\xi_3\bar{\xi}_3+3\xi_3\bar{\xi}_1+3\bar{\xi}_3\xi_1,\\
F_6 &= \quad(\alpha-1)\xi_1\bar{\xi}_1+(\alpha+2)\xi_2\bar{\xi}_2-(2\alpha+1)\xi_3\bar{\xi}_3-3\xi_1\bar{\xi}_2-3\bar{\xi}_1\xi_2.
\end{aligned}$$

The fifteen forms conjugate to $2\xi_1\bar{\xi}_2+2\bar{\xi}_1\xi_2$ are $F_{ik}=-F_{ki}=F_i-F_k$. There are twenty relations between these forms $F_{ij}+F_{jk}+F_{ki}\equiv 0$.

It is found that the Hermitian forms F_{12}, F_{54}, F_{16}, F_{34} and F_{52} meet the real plane in the sides of the triangle whose sides are $x+\alpha y-\alpha^2=0$, $x=0$ and $y=0$.

For any point which lies in this triangle, the signs of F_{12} and F_{54} are both $-$, while those of F_{16}, F_{34} and F_{52} are all $+$.

Taking F_{12} and F_{54} as negative and the other three forms as positive, and making use of the twenty relations between the forms, it is seen that the signs of the remaining ten forms are determined. Therefore, none of these ten forms can *cross* the *region* in S_4 which is determined by the five forms under consideration. Since for this region $F_3>F_4>F_5>F_2>F_1>F_6$, it can be written [345216].

Under G_{60} the region [345216] has sixty conjugate regions. These meet the real plane in distinct triangles, the fundamental regions, for the group, in the plane.

The question arises into how many such regions is S_4 divided by the forms F_{ik}?

For any point in S_4 not on an F_{ik} the values of the F_i are all distinct, and, since these six values admit at most seven hundred and twenty permutations, the forms F_{ik} have at most seven hundred and twenty arrangements of sign.

Under G_{60} the seven hundred and twenty value systems of the F_i divide into twelve conjugate sets of sixty each, and if one value system is taken from each set we obtain a fundamental region. As an example of such a fundamental region we give the twelve value systems determined by the inequalities $F_3 > F_4 > F_1$, $F_4 > F_6$, $F_5 > F_2 > F_1$ and $F_2 > F_6$. No two of these are conjugates under G_{60} and therefore they comprise a fundamental region in S_4 for the group, except for the points which reduce one or more of the F_{ik} to zero.

On the Representation of Functions in Series of the Form $\Sigma c_n g(x+n)$.*

BY R. D. CARMICHAEL.

Introduction.

The most important functions defined by the Ω- and $\bar{\Omega}$-series, whose properties I have investigated in previous memoirs,† are doubtless those which have in a half-plane a Poincaré asymptotic representation ‡ in descending power series; and, in particular, those in which the series depend on a defining function $g(x)$ which has the asymptotic form §

$$g(x) \sim x^{\mu-x} e^{\alpha+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\right), \tag{1}$$

valid in a sector V including the positive axis of reals in its interior, and which is analytic in V for sufficiently large values of $|x|$. We shall now use the symbol $S(x)$ to denote the series

$$S(x)=\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}. \tag{2}$$

It will be observed that $S(x)$ belongs to the class of series denoted by $\bar{\Omega}(x)$ in the preceding papers, and also to the more special class denoted by $\bar{\omega}(x)$.

In the present paper I make a contribution towards solving the problem ‖ of representing given functions in the form of series $S(x)$ depending on a

* Presented to the American Mathematical Society (at Chicago), April 7, 1917.

† *Transactions American Mathematical Society*, Vol. XVII (1916), pp. 207–232; AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIX (1917), pp. 385–403. These papers will be referred to by the numbers I and II, respectively.

‡ Compare II, especially §§ 4 and 5.

§ It is convenient to introduce here a slight change in notation; it is one, however, which can cause no confusion.

‖ That this includes the fundamental problem of representing given functions in the form of series dependent on functions defined by linear homogeneous difference equations may be seen from the following considerations. A fundamental property of the leading functions $G(x)$ defined by such equations is that expressed in the asymptotic relation

$$G(x) \sim x^{ix} e^{\beta x}\left(1+\frac{b_1}{x}+\frac{b_2}{x^2}+\ldots\right)$$

valid in a right half-plane, i being an integer and $\beta, b_1, b_2, \ldots$ being constants. (Compare Birkhoff, *Transactions American Mathematical Society*, Vol. XII (1911), pp. 243–284; and Carmichael, *ibid.*, pp.

given defining function $g(x)$. In order to render this problem amenable to simple methods it is necessary to place further restrictions on $g(x)$. We have seen (II, § 5) that

$$\frac{g(x+n)}{g(x)} \sim x^{-n}\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\frac{\beta_{2n}}{x^2}+\dots\right), \quad \beta_{0n}=e^{(\beta-1)n}, \tag{3}$$

where $\beta_{0n}, \beta_{1n}, \beta_{2n}, \dots$ are a set of numbers independent of x. From this it follows that we have an asymptotic relation of the form

$$\frac{g(x)}{g(x+n)} \sim x^{n}\left(\gamma_{0n}+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\dots\right). \tag{4}$$

Some of the restrictions mentioned are stated most conveniently in terms of the coefficients γ. They appear in Section 4. Others appear in Section 5.

In § 1 I derive, in simple form, a necessary and sufficient condition on the coefficients c_n implying the convergence of $S(x)$ in an appropriately determined right half-plane. In § 2 some remarks are made on the order of increase of the coefficients in certain Poincaré asymptotic series. In § 3 the fundamental relations between these coefficients and the coefficients c_n of the associated series $S(x)$ are determined. In § 4 I construct, in a particular manner, functions having given Poincaré asymptotic representations of a certain type, and incidentally point out some fundamentally important instances of the series $S(x)$ which have occurred in the recent literature. Finally, in § 5, I show how to transform the Borel integral sum of a divergent series into a series $S(x)$ and also into a certain natural generalization of such series, and indicate some wide ranges of applicability of these results.

§ 1. *Order of Increase of Coefficients c_n in a Converging Series $S(x)$.*

Let $S(x)$ be a series of the form (2) which converges at every non-exceptional point in some half-plane; that is, one whose convergence number is not $-\infty$. Then, from the corollary to Theorem XII, in Memoir I, it follows that a real constant r_1 (finite or equal to $-\infty$) exists such that

$$\limsup_{\xi=\infty} \frac{\log \Sigma_\xi |c_\nu g(\nu)|}{\xi} = r_1, \tag{5}$$

99–134, and AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1916), pp. 163–182). In case $i=-1$ the function $G(x)$ itself is of the same type as the function $g(x)$ in the text. In general a suitable value of $g(x)$ may be defined in terms of $G(x)$ by the relation

$$g(x)=\frac{G(x)}{[\Gamma(x)]^{i+1}},$$

or by any one of several similar expressions which readily come to mind. Thus, for every function of the type $G(x)$ there exist corresponding functions of the type $g(x)$ suitable for use in the formation of series $S(x)$.

where $\Sigma_\xi \beta_\nu$ stands for the sum of all β_ν whose suffix ν satisfies the relation

$$e^{E(\xi)} \leqq \nu < e^\xi, \tag{6}$$

$E(\xi)$ denoting the greatest integer not greater than ξ. From the last inequality we see that

$$\xi < E(\xi)+1 \leqq 1+\log \nu$$

for every positive value of ξ and corresponding value of ν. Hence, from (5), it follows that a constant r_2 exists such that

$$\log |c_\nu g(\nu)| < r_2(1+\log \nu), \qquad \nu > 0.$$

Thence it follows readily that a constant r exists such that

$$|c_\nu g(\nu)| < \nu^r, \qquad \nu \geqq 2. \tag{7}$$

Relation (7) states a condition which is necessary if $S(x)$ is to converge at every non-exceptional point in some half-plane.

It may also be shown that (7) states a condition sufficient to ensure that $S(x)$ thus converges in some half-plane. For from (7) it is clear that

$$\Sigma_\xi |c_\nu g(\nu)| < \nu^\rho e^\xi,$$

where ν and ξ are related as in (6), and ρ is a positive number not less than r. From this one sees readily that the superior limit in the first member of (5) has a value different from $+\infty$. This, in connection with the corollary to Theorem XII, in Memoir I, at once yields the conclusion that $S(x)$ converges at every non-exceptional point in some half-plane.

Thus we are led to the following theorem:*

Theorem I. *A necessary and sufficient condition that the series $S(x)$ in (2) shall converge for every non-exceptional value of x in some right half-plane is that a constant r exists such that*

$$|c_\nu g(\nu)| < \nu^r, \qquad \nu \geqq 2.$$

In view of the asymptotic form of $g(\nu)$ this theorem is seen to be equivalent to the following corollary:

Corollary: *A necessary and sufficient condition that the series $S(x)$ in (2) shall converge for every non-exceptional value of x in some right half-plane is that a constant r exists such that*

$$|c_\nu| < \nu^\nu e^{-\nu R(\beta)} \nu^r, \qquad \nu \geqq 2.$$

* By exactly the same argumentation one is led to precisely the same necessary and sufficient condition for the similar convergence of the series $\Omega(x)$ and $\bar{\Omega}(x)$ in the case when $k=1$ and $m=0$ or 1. The corollary to Theorem XII, in Memoir I, yields immediately the corresponding result for the case when k and m do not satisfy the conditions just stated.

It is easy to see that an equivalent statement is obtained if one replaces the inequality in this corollary by the following:

$$|c_\nu| < \nu!\, e^{\nu[1-R(\beta)]}\nu^r, \qquad \nu \geqq 2.$$

To prove this, one has only to employ the well-known fact that

$$(\nu+1)!\,\nu^{-\nu}e^{\nu}\nu^{\frac{1}{2}}$$

approaches a finite limit different from zero as ν approaches infinity.

§ 2. *Order of Increase of the Coefficients in the Poincaré Asymptotic Representation of $S(x)$.*

We have seen (II, Theorem II and § 5) that a function $S(x)$, defined by the series in (2), has a Poincaré asymptotic representation of the form

$$S(x) \sim \beta_0 + \frac{\beta_1}{x} + \frac{\beta_2}{x^2} + \ldots . \tag{8}$$

valid in that part of V which lies in the half-plane $R(x) \geqq -\lambda_1$, where λ_1 is less than the convergence number λ of $S(x)$, and that the coefficients β_ν are given by the formulae

$$\beta_\nu = c_0\beta_{\nu 0} + c_1\beta_{\nu-1,1} + c_2\beta_{\nu-2,2} + \ldots + c_\nu\beta_{0\nu}, \qquad \nu = 0, 1, 2, \ldots . \tag{9}$$

The order of increase of $|\beta_\nu|$ with respect to ν evidently depends upon the β_{ij} quite as much as upon the c_i. In case the β_{ij} satisfy the restrictive condition

$$|\beta_{\nu-k,k}| < Me^{kR(\beta)}\nu^{\nu+\sigma}k^{-k}, \tag{10}$$

it is easy to see (by aid of the corollary to Theorem I) that we have

$$|\beta_\nu| < \nu^\nu \nu^s, \qquad \nu \geqq 2, \tag{11}$$

s being an appropriately chosen quantity independent of ν.

For the special case in which $g(x+1)/g(x)$ is analytic at $x=\infty$, other inequalities similar to (10) and (11) may be obtained. We write

$$\frac{g(x+1)}{g(x)} \equiv \rho(x) = \frac{\rho_1}{x} + \frac{\rho_2}{x^2} + \frac{\rho_3}{x^3} + \ldots .$$

Since this series converges for some x, a positive quantity ρ exists such that

$$|\rho_r| < \rho^r.$$

We take ρ to be greater than unity. We may write,

$$\frac{g(x+n)}{g(x)} \equiv \rho(x)\rho(x+1)\ldots\rho(x+n-1) = \prod_{i=0}^{n-1}\left[\frac{\rho_1}{x+i} + \frac{\rho_2}{(x+i)^2} + \ldots\right]$$
$$= \prod_{i=0}^{n-1}\left[\frac{\rho_1}{x} + \frac{\rho_2 - \rho_1 i}{x^2} + \frac{\rho_3 - 2\rho_2 i + \rho_1 i^2}{x^3} + \ldots\right].$$

If this last product is expanded in powers of $1/x$, it is easy to see that the coefficients in the resulting series are less in absolute value than the coefficients in the similar expansion of the product

$$\prod_{i=0}^{n-1}\left[\frac{\rho}{x}+\frac{\rho(\rho+i)}{x^2}+\frac{\rho(\rho+i)^2}{x^3}+\frac{\rho(\rho+i)^3}{x^4}+\dots\right].$$

These coefficients are not decreased if in this product $\rho(\rho+i)^{m-1}$ is replaced by $(\rho+n)^m$. Thence, by the multinomial theorem, it follows readily that

$$|\beta_{\nu-n,n}|<(n+\rho)^\nu\Sigma\frac{(i_1+i_2+\dots+i_n)!}{i_1!\,i_2!\dots i_n!},$$

where the summation is taken subject to the condition

$$i_1+2i_2+3i_3+\dots+ni_n=\nu;$$

whence it follows that

$$|\beta_{\nu-n,n}|<\nu^n(n+\rho)^\nu. \tag{12}$$

Thence, through (9), and the corollary to Theorem I, we see that

$$|\beta_\nu|<\nu^{3\nu}e^{-\nu R(\beta)}\nu^t, \qquad \nu\geqq 2, \tag{13}$$

t being an appropriately chosen real quantity.

§ 3. *Evaluation of the Constants c_ν in Terms of the Constants β_ν.*

From (9) it follows that the constants c_ν may be expressed directly in terms of the constants β_ν. It is more convenient for our purposes, however, to proceed as follows: In equation (4) replace n by $\nu-n$, where $n<\nu$, and then replace x by $x+n$. Thus we have

$$\frac{g(x+n)}{g(x+\nu)}\sim(x+n)^{\nu-n}\left(\gamma_{0,\nu-n}+\frac{\gamma_{1,\nu-n}}{x+n}+\frac{\gamma_{2,\nu-n}}{(x+n)^2}+\dots\right). \tag{14}$$

Then from

$$\frac{g(x)}{g(x+\nu)}\left(\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\dots\right)\sim\sum_{n=0}^{\infty}c_n\frac{g(x+n)}{g(x+\nu)},$$

by means of (4) and (14), we have

$$\gamma_{0\nu}\beta_{\nu+1}+\gamma_{1\nu}\beta_\nu+\gamma_{2\nu}\beta_{\nu-1}+\dots+\gamma_{\nu+1,\nu}\beta_0=\frac{1}{\gamma_{01}}c_{\nu+1}+\gamma_{21}c_{\nu-1}$$
$$+\gamma_{32}c_{\nu-2}+\dots+\gamma_{\nu+1,\nu}c_0,$$

on equating the coefficients of $1/x$ in the expanded asymptotic forms of the two members. Now it is easy to see that

$$c_0=\beta_0, \quad c_1=\beta_1. \tag{15}$$

Hence, if we write

$$\eta_\nu=\gamma_{0\nu}\beta_{\nu+1}+\gamma_{1\nu}\beta_\nu+\gamma_{2\nu}\beta_{\nu-1}+\ldots+\gamma_{\nu\nu}\beta_1,\ \nu>0;\qquad \eta_0=\beta_1, \tag{16}$$

we have

$$\frac{1}{\gamma_{01}}c_{\nu+1}+\gamma_{21}c_{\nu-1}+\gamma_{32}c_{\nu-2}+\ldots+\gamma_{\nu,\nu-1}c_1=\eta_\nu,\qquad \nu=1,2,\ldots.$$

Combining this system with the second equation in (15), and solving for $c_{\nu+1}$ we have

$$c_{\nu+1}=\gamma_{01}\eta_\nu+\gamma_{01}^3\eta_{\nu-2}\begin{vmatrix} 0 & \gamma_{21} \\ \frac{1}{\gamma_{01}} & 0 \end{vmatrix}-\gamma_{01}^4\eta_{\nu-3}\begin{vmatrix} 0 & \gamma_{21} & \gamma_{32} \\ \frac{1}{\gamma_{01}} & 0 & \gamma_{21} \\ 0 & \frac{1}{\gamma_{01}} & 0 \end{vmatrix}$$

$$+\gamma_{01}^5\eta_{\nu-4}\begin{vmatrix} 0 & \gamma_{21} & \gamma_{32} & \gamma_{43} \\ \frac{1}{\gamma_{01}} & 0 & \gamma_{21} & \gamma_{32} \\ 0 & \frac{1}{\gamma_{01}} & 0 & \gamma_{21} \\ 0 & 0 & \frac{1}{\gamma_{01}} & 0 \end{vmatrix}-\ldots,\qquad \nu=1,2,3,\ldots, \tag{17}$$

the expansion ending with the term containing η_0. Here γ_{01} has the value

$$\gamma_{01}=e^{1-\beta}.$$

We shall use the symbol Δ_k for the determinant of order k in the second member of equation (17).

§ 4. *Construction of Functions Having Given Poincaré Asymptotic Representations of a Certain Type.*

In this section we show how to construct a function $f(x)$ having a given Poincaré asymptotic representation

$$f(x)\sim\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\ldots, \tag{18}$$

of a certain type. We confine our attention to the case in which the function $g(x)$ and the constants $\beta_0, \beta_1, \beta_2, \ldots$ are jointly subject to the condition that positive quantities M and σ, both independent of ν, exist such that

$$|\eta_\nu|<M\nu^\nu e^{-\nu R(\beta)}\nu^\sigma,\qquad \nu=1,2,3,\ldots, \tag{19}$$

η_ν having the definition given in (16). Moreover, we suppose that $g(x)$ is such that a positive quantity M_1, and a real non-negative quantity σ_1, both independent of k, exist such that

$$|\Delta_k|<M_1k^ke^{-k}k^{\sigma_1}, \tag{20}$$

where Δ_k denotes the determinant defined at the end of Section 3. Our central theorem here is the following:

THEOREM II. *Let* $\beta_0, \beta_1, \beta_2, \ldots,$ *be a given set of constants, and let* $g(x)$ *be an associated function such that relations* (19) *and* (20) *are satisfied. Then, if constants* $c_0, c_1, c_2, \ldots$ *are determined in terms of* $\beta_0, \beta_1, \beta_2, \ldots,$ *by means of equations* (15) *and* (17), *the series*

$$\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)} \tag{21}$$

converges absolutely at all non-exceptional points in the half-plane $R(x) > \tau+1$, *where* τ *is the greater of the two quantities* 0 *and* $\sigma+\sigma_1+R(\mu)$; *and the function* $f(x)$ *represented by this series satisfies the Poincaré asymptotic relation* (18), *the latter being valid in the greatest region in the* x*-plane which is common to* V *and the half-plane* $R(x) \geqq \tau+1+\varepsilon$, ε *being any positive constant.*

We prove first that part of the conclusion which relates to the convergence of the series in (21). From relations (17), (19), (20) we see readily that

$$|c_{\nu+1}| < M_2 \nu^\nu e^{-\nu R(\beta)} \nu^{\sigma+\sigma_1+1}, \tag{22}$$

where M_2 is a quantity independent of ν. From the corollary to Theorem I it follows now that the series in (21) is convergent in an appropriately chosen right half-plane. Moreover, the last inequality affords partial information as to the position of the line of absolute convergence. This may be shown as follows: From the corollary to Theorem XII, in Memoir I, we see that the abscissa μ of absolute convergence of the series in (21) is given by the relation

$$\mu = -\limsup_{\xi=\infty} \frac{\log \Sigma_\xi |c_\nu g(\nu)|}{\xi},$$

where $\Sigma_\xi \zeta_\nu$ stands for the sum of all ζ_ν whose suffix ν satisfies the relation

$$e^{E(\xi)} \leqq \nu < e^\xi.$$

From (22) and the asymptotic form of $g(\nu)$ it is easy to see that

$$|c_\nu g(\nu)| < M_3 \nu^{\sigma+\sigma_1+R(\mu)}, \qquad \nu = 2, 3, \ldots,$$

M_3 being an appropriately chosen constant. Hence we have

$$\Sigma_\xi |c_\nu g(\nu)| < M_3 \sum_{\nu=1}^{\eta_\xi} \nu^{\sigma+\sigma_1+R(\mu)} \leqq M_3 \eta_\xi^{\tau+1},$$

where η_ξ is the greatest value of ν, such that $\log \nu < \xi$ and τ is the greater of the quantities 0 and $\sigma+\sigma_1+R(\mu)$. Hence,

$$\mu \geqq -\limsup_{\xi=\infty} \frac{(\tau+1)\log \eta_\xi + \log M_3}{\xi} = -(\tau+1).$$

This completes the proof of the part of the theorem which refers to the convergence of the series in (21).

The portion of the conclusion of the foregoing theorem which relates to the asymptotic character of $f(x)$ is an immediate consequence of Theorem II, of Memoir II, and the convergence properties just established.

The foregoing theorem states conditions to ensure the convergence of the series in (21). A central part of these conditions is contained in relations (19). If $g(x)$ satisfies conditions associated with (1), and if, moreover, $g(x)/g(x+\nu)$ is a polynomial when ν is sufficiently large, then from (17), and the corollary to Theorem I, it is easy to see that *relation* (19) *affords a necessary and sufficient condition for the convergence of* (21); and hence for the construction, by means of a series (21), of a function $f(x)$ having the given Poincaré asymptotic representation (18). It is not difficult to determine other classes of functions $g(x)$ for which (19) plays a like rôle in respect to necessary and sufficient conditions.

An important special use of the result contained in Theorem II should be pointed out. In many investigations, particularly in the theory of differential equations and of difference equations, one is led to divergent power series when one seeks to obtain a suitable representation of a function which is to be determined. If for an appropriately chosen function $g(x)$ the coefficients $\beta_0, \beta_2, \beta_3, \ldots$, in the diverging power series thus arising satisfy the conditions imposed by relations (19), then it is clear that a suitable modification of the computations will enable one to obtain a convergent expansion (21) in place of the diverging power series. It may be anticipated with considerable confidence that this converging expansion will serve to define such a function as one is seeking. It is my intention later to present such applications of Theorem II as are here indicated. For the special class of factorial series, important applications of this character have already been made in the theory both of differential equations* and of difference equations,† and in more general aspects of the theory of functions.‡

We shall next point out some simple conditions which are sufficient to ensure that relations (19) and (20) shall be satisfied. In the treatment of these conditions we shall have need of Hadamard's fundamental theorem§ concerning an upper bound to the absolute value of a determinant. This theorem may be stated as follows:

---

* Horn, *Mathematische Annalen*, Vol. LXXI (1912), pp. 510–532.

† Nörlund, *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXV (1913), pp. 177–216.

‡ Watson, *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXIV (1912), pp. 41–48.

§ Hadamard, *Bulletin des Sciences Mathématiques* (Darboux), Vol. XVII (1893), pp. 240–246.

If Δ is a determinant of order n in which a_{ij} is the element in the *i-th* row and the *j-th* column, then

$$|\Delta| \leqq \sqrt{r_1 r_2 \ldots . r_n}, \quad |\Delta| \leqq \sqrt{\rho_1 \rho_2 \ldots . \rho_n},$$

where

$$r_i = \sum_{j=1}^{n} |a_{ij}|^2, \quad \rho_i = \sum_{j=1}^{n} |a_{ji}|^2.$$

With this theorem in hand we may readily determine conditions implying relation (20). Thus, if $\gamma_{k+1,k}$ satisfies the condition

$$|\gamma_{k+1,k}| \leqq k^{\frac{1}{2}}, \qquad k = K, K+1, K+2, \ldots, \tag{23}$$

where K is a fixed integer, it may easily be shown that (20) is satisfied. For, from the first inequality in Hadamard's theorem above, we see that

$$|\Delta_k| < \bar{M} \cdot k! = \bar{M} \cdot \Gamma(k+1),$$

$\bar{M}$ being an appropriately chosen quantity independent of k. By means of the well-known asymptotic formula for $\Gamma(x)$, namely,

$$\Gamma(x) \sim x^x e^{-x} x^{-\frac{1}{2}} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \ldots . \right),$$

it is now easy to see that (20) is satisfied provided that M_1 and σ_1 are given appropriate values. Hence, relation (23) expresses a condition sufficient *to ensure that a relation of the form* (20) *is satisfied.*

In order to determine simple conditions ensuring that (19) is satisfied we observe that the asymptotic relation (1) implies that the quantity

$$\frac{g(\rho+\nu)}{g(\rho)} \nu^\nu e^{-\beta\nu} \nu^{\rho-}$$

approaches a finite value different from zero as n approaches infinity, provided that $x=\rho$ is a point at which $g(x)$ is analytic and different from zero. From this it follows at once that a relation of the form (19) is satisfied, provided that a positive quantity M' and a non-negative real quantity σ', both independent of ν, exist such that

$$|\eta_\nu| < M' \left| \frac{g(\rho)}{g(\rho+\nu)} \right| \nu^{\sigma'}. \tag{24}$$

Let us consider now the special case in which the series in (18) converges when $|x|$ is sufficiently large. Then through (16) we see that a positive number ρ exists such that

$$\frac{1}{\rho} |\eta_\nu| \leq |\gamma_{0\nu}| \rho^\nu + |\gamma_{1\nu}| \rho^{\nu-1} + |\gamma_{2\nu}| \rho^{\nu-2} + \ldots . + |\gamma_{\nu-1,\nu}| \rho + |\gamma_{\nu\nu}|,$$
$$\nu = 1, 2, 3, \ldots . .$$

Comparing this with (4) we see that condition (24) is obviously satisfied in the special case when the quantities $\gamma_{0n}, \gamma_{1n}, \gamma_{2n}, \ldots.$ are all positive or zero for every positive integral value of n greater than some given value N. Other simple conditions under which (24) is satisfied will readily occur to the reader.

Again, it may be seen that a relation of the form (19) is satisfied in case a positive quantity M and a non-negative real quantity σ exist such that

$$|\gamma_{k\nu}\beta_{\nu-k+1}| < M\nu^{\nu}e^{-\nu R(\beta)}\nu^{\sigma-1} \tag{25}$$

for every ν and corresponding k not greater than ν. For determining whether (25) is satisfied one may make use of the relation

$$\gamma_{k\nu} = \frac{1}{2\pi i}\int_C \frac{f_\nu(x)\,dx}{x^{\nu-k+1}}, \tag{26}$$

where

$$f_\nu(x) = \gamma_{0\nu}x^{\nu} + \gamma_{1\nu}x^{\nu-1} + \ldots. + \gamma_{\nu\nu}$$

and C is a circle about the point $x=0$ as a center. From this it follows at once that

$$|\gamma_{k\nu}|\rho^{\nu-k} \leqq \frac{1}{2\pi\rho}M_\rho[f_\nu(x)], \tag{27}$$

where $M_\rho[f_\nu(x)]$ denotes the maximum value of $|f_\nu(x)|$ on the circle of radius ρ about the point $x=0$ as a center. There are large classes of cases in which it may readily be shown that relation (25) is implied by relation (27).

Let us consider the application of these results in the case of factorial series. Here $g(x) = 1/\Gamma(x)$; and we have

$$\frac{g(x)}{g(x+n)} = x(x+1)(x+2)\ldots.(x+n).$$

It is obvious that the quantities γ_{sn} are all positive or zero, and that they are zero in case $s \geqq n$. Moreover,

$$\gamma_{0\nu}\rho^{\nu} + \gamma_{1\nu}\rho^{\nu-1} + \ldots. + \gamma_{\nu\nu} = \frac{g(\rho)}{g(\rho+\nu)}.$$

Hence, conditions (23) and (24) are satisfied in case (18) converges for sufficiently large values of $|x|$, say for $|x| \geqq R$. Moreover, for this special case it is clear that we may take $\sigma_1 = 0$ and $\sigma = R - \mu$ and that μ now has the value $\frac{1}{2}$. Then it is easy to see that the line of absolute convergence of the series in (21), in this case cuts the axis of reals at a point not further to the right than a unit's distance to the right of the rightmost point of the circle of convergence of the power series in $1/x$ by which $f(x)$ may be represented.

Again, if we take for $g(x)$ the value

$$g(x)=\frac{1}{a^x\Gamma\left(x+\frac{b}{a}\right)}, \tag{28}$$

we have

$$\frac{g(x)}{g(x+1)}=ax+b.$$

Then, as in the preceding case, it is easy to see that the conditions of Theorem II are satisfied provided that (18) converges for sufficiently large $|x|$ and a is positive while b is positive or zero. If we put $b=0$, series (1) obviously takes the special form

$$S_1(x)=c_0+\sum_{n=1}^{\infty}\frac{c_n}{ax(ax+a)(ax+2a)\ldots.(ax+\overline{n-1}a)}.$$

Replacing ax by z, we may write this in the form

$$\bar{S}_a(z)=c_0+\sum_{n=1}^{\infty}\frac{c_n}{z(z+a)(z+2a)\ldots.(z+[n-1]a)}. \tag{29}$$

Series of this class play a leading rôle in Nörlund's fundamental paper on factorial series to which reference has already been made.

Again, if we put Mz for x and $\alpha+1$ for b/a in the function $g(x)$ of equation (28) the corresponding series $S(x)$ is transformed into the form

$$b_0+\frac{b_1}{Mz+\alpha+1}+\frac{b_2}{(Mz+\alpha+1)(Mz+\alpha+2)} \\ +\frac{b_3}{(Mz+\alpha+1)(Mz+\alpha+2)(Mz+\alpha+3)}+\ldots., \tag{30}$$

a series which plays the leading rôle in the important paper of Watson's referred to above. Watson exhibits a large class of functions expansible in converging series (30).

One may easily construct many other particular functions $g(x)$ satisfying those hypotheses of Theorem II which relate to $g(x)$ provided that the associated series (18) converges for sufficiently large $|x|$. Some of the most interesting of such functions $g(x)$ are readily expressible in terms of the gamma function. Such a one, for instance, is the function $g(x)$,

$$g(x)=\frac{\Gamma(x+1)}{\Gamma(x)\Gamma(x+3)}. \tag{31}$$

Here we have

$$\frac{g(x)}{g(x+1)}=\frac{x(x+3)}{x+1}.$$

Employing the relation

$$\frac{g(x)}{g(x+n)}=\frac{g(x)}{g(x+1)}\cdot\frac{g(x+1)}{g(x+2)}\cdots\frac{g(x+n-1)}{g(x+n)},$$

it is thence easy to see that $g(x)/g(x+n)$ is a polynomial with non-negative real coefficients provided that n is sufficiently large. Hence, relations (23) and (24) are satisfied.

In a similar way one may treat the function

$$g(x)=\frac{\Gamma(x+1)\Gamma(x+3)}{\Gamma(x)\Gamma(x+2)\Gamma(x+4)}. \tag{32}$$

It is clear that one may thus form an indefinitely great number of similar functions $g(x)$ such that in each case $g(x)/g(x+1)$ is a rational function of x, and $g(x)/g(x+n)$ is a polynomial with non-negative real coefficients provided that n is sufficiently large.

§ 5. *Representation of Given Functions in the Form of Convergent Series $S(x)$.*

By means of Borel's method of summation of series (in general divergent) we shall now show how to represent functions of a certain important class in the form of convergent series $S(x)$. The functions treated are those obtained by taking the sum of a series of the form

$$\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\frac{\beta_3}{x^3}+\ldots. \tag{33}$$

by means of Borel's integral method of summation. Denoting the Borel integral sum of the series in (33) by $f(x)$, we have by definition:*

$$f(x)=\int_0^\infty xe^{-tx}\phi(t)\,dt, \tag{34}$$

where

$$\phi(t)=\beta_0+\beta_1 t+\frac{\beta_2 t^2}{2!}+\frac{\beta_3 t^3}{3!}+\ldots+\frac{\beta_n t^n}{n!}+\ldots. \tag{35}$$

We assume† that the series in (35) is convergent for all values of t, and that the function $\phi(t)$ defined by it is such that the integral in (34) exists. We say then that the series (34) is summable to the sum $f(x)$.

In this section we shall suppose† also that the asymptotic series in (3) is summable in the sense of Borel, so that we have

$$\frac{g(x+n)}{g(x)}=\int_0^\infty xe^{-tx}\phi_n(t)\,dt, \qquad n=1, 2, 3, \ldots, \tag{36}$$

where

$$\phi_n(t)=\frac{\beta_{0n}t^n}{n!}+\frac{\beta_{1n}t^{n+1}}{(n+1)!}+\frac{\beta_{2n}t^{n+2}}{(n+2)!}+\ldots. \tag{37}$$

The series in (37) we shall take to be convergent for all values of t.

---

* Borel, "Leçons sur les séries divergentes," p. 103 ff. We have made certain obvious reductions so as to obtain the form convenient to use with descending series (33) rather than with ascending series.

† The restrictions made here are stronger than are essential to the argument. They may be weakened in accordance with certain general considerations mentioned by Borel (*loc. cit.*, p. 99).

Of the function $f(x)$ defined in (34) it is easy to obtain a *formal* expansion in series $S(x)$. For this purpose we note that by means of equation (9) it is easy to establish the formal relation

$$\beta_0+\beta_1 t+\ldots+\frac{\beta_n t^n}{n!}+\ldots=\sum_{n=0}^{\infty} c_n \sum_{\nu=n}^{\infty} \beta_{\nu-n,n}\frac{t^\nu}{\nu!}\equiv\sum_{n=0}^{\infty} c_n\phi_n(t). \tag{38}$$

Replacing $\phi(t)$ in (34) by the second member of the last equation we have the formal relation

$$f(x)=\int_0^\infty\left(\sum_{n=0}^{\infty} c_n x e^{-tx}\phi_n(t)\right)dt. \tag{39}$$

If, still proceeding formally, we integrate term by term the series denoted by the summation in (39) and make use of (36), we have

$$f(x)=\sum_{n=0}^{\infty} c_n\frac{g(x+n)}{g(x)}. \tag{40}$$

Hence,

THEOREM III. *In all cases in which* (36) *and* (38) *are valid relations, and the series denoted by the outer summation in* (39) *is term by term integrable from zero to infinity, the Borel sum* $f(x)$ *of series* (33) *is represented in* (40) *in the form of a converging series* $S(x)$.

This result is capable of a ready generalization as follows: From (34) we have

$$f(ax)=\int_0^\infty axe^{-atx}\phi(t)\,dt=\int_0^\infty xe^{-tx}\psi(t)\,dt, \tag{41}$$

where

$$\psi(t)=\beta_0+\frac{\beta_1}{a}t+\frac{\beta_2}{a^2}t^2+\frac{\beta_3}{a^3}t^3+\ldots\ldots$$

Analogous to (9) we now form the equations

$$\frac{\beta_\nu}{a^\nu}=\bar{c}_0\beta_{\nu 0}+\bar{c}_1\beta_{\nu-1,1}+\bar{c}_2\beta_{\nu-2,2}+\ldots+\bar{c}_\nu\beta_{0\nu},\qquad \nu=0,1,2,\ldots,$$

thus introducing the quantities $\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots$. Thence proceeding formally, we have

$$\beta_0+\frac{\beta_1}{a}t+\frac{\beta_2}{a^2}t^2+\ldots=\sum_{n=0}^{\infty}\bar{c}_n\sum_{\nu=n}^{\infty}\beta_{\nu-n,n}\frac{t^\nu}{\nu!}\equiv\sum_{n=0}^{\infty}\bar{c}_n\phi_n(t); \tag{42}$$

whence, as before, we obtain

$$f(ax)=\int_0^\infty\left(\sum_{n=0}^{\infty}\bar{c}_n xe^{-tx}\phi_n(t)\right)dt. \tag{43}$$

Integrating term by term, we have

$$f(ax)=\sum_{n=0}^{\infty}\bar{c}_n\frac{g(x+n)}{g(x)};$$

or

$$f(x)=\sum_{n=0}^{\infty}\bar{c}_n\frac{g(x/a+n)}{g(x/a)}. \tag{44}$$

We are thus lead to the following result:

THEOREM IV. *In all cases in which* (36) *and* (42) *are valid relations and the series denoted by the summation in* (43) *is term by term integrable from zero to infinity, the Borel sum* $f(x)$ *of series* (33) *is represented in* (44) *in the form of a converging series into which a series* $S(x)$ *is readily transformed.*

If we take for $g(x)$ the particular value

$$g(x)=\frac{1}{a^x\Gamma(x)},$$

then relation (44) takes the special form

$$f(x)=\bar{c}_0+\frac{\bar{c}_1}{x}+\frac{\bar{c}_2}{x(x+a)}+\frac{\bar{c}_3}{x(x+a)(x+2a)}+\frac{\bar{c}_4}{x(x+a)(x+2a)(x+3a)}+\cdots\cdot \tag{45}$$

That is to say, the Borel sum $f(x)$ of (33) is always represented formally by the series in (45). In case (33) is absolutely and uniformly summable by the integral method of Borel to the sum $f(x)$, then Nörlund (*loc. cit.*, p. 379) has shown that $f(x)$ has an actual convergent expansion (45) in case a is a sufficiently large positive number dependent upon $f(x)$. Hence the formal result in (45), and therefore the more general one in (44), has a wide range of useful applicability.

For special functions $g(x)$ (of which that in the preceding paragraph is an example) and corresponding classes of functions $f(x)$ it is possible, as we have just seen, to state more explicit and precise results than those obtained in Theorems III and IV; but it seems to be difficult to render these theorems more precise without further restrictions on $g(x)$. I propose as an important problem the determination of classes of functions $g(x)$, and corresponding classes of functions $f(x)$, such that the representations (40) and (44) are valid, either separately or simultaneously.

UNIVERSITY OF ILLINOIS, *February*, 1917.

Transformations of Planar Nets.

BY LUTHER PFAHLER EISENHART.

1. **Introduction.** When a surface is referred to a conjugate system of curves, u and v being the parameters, the four homogeneous point coordinates, $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $x^{(4)}$, are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v}=a\frac{\partial\theta}{\partial u}+b\frac{\partial\theta}{\partial v}+c\theta, \tag{1}$$

where a, b, c are functions of u and v. We say that the parametric curves u=const., v=const. form a net $N(x)$, x indicating any of the four coordinates, and call (1) *the point equation* of $N(x)$.

If θ is any other solution of (1), the equations

$$\frac{\partial x_1}{\partial u}=h\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right),\quad \frac{\partial x_1}{\partial v}=l\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right), \tag{2}$$

satisfy the condition of integrability, provided that h and l are any pair of solutions * of

$$\frac{\partial h}{\partial v}=\left(\frac{\partial \log\theta}{\partial v}-a\right)(h-l),\quad \frac{\partial l}{\partial u}=\left(\frac{\partial \log\theta}{\partial u}-b\right)(l-h). \tag{3}$$

Moreover, the four functions x_1 obtained by quadratures from (2) when x takes on the four values of the coordinates of $N(x)$ are solutions of the same equation

$$\frac{\partial^2\theta_1}{\partial u\partial v}+\frac{l}{h}\left(\frac{\partial \log\theta}{\partial v}-a\right)\frac{\partial\theta_1}{\partial u}+\frac{h}{l}\left(\frac{\partial \log\theta}{\partial u}-b\right)\frac{\partial\theta_1}{\partial v}=0. \tag{4}$$

Hence these four quantities x_1 are the homogeneous point coordinates of a net $N_1(x_1)$.

The points F_1 and F_2, with respective homogeneous coordinates

$$\theta x_1-hx,\quad \theta x_1-lx, \tag{5}$$

lie on the line joining corresponding points, M and M_1, of the nets N and N_1, and are such that as u or v varies, F_1 or F_2, respectively, moves tangentially

---

* We exclude the case $h=l=$const., since in this case the two surfaces are homothetic with respect to the origin.

to the line. Hence the ruled surfaces $u=\text{const.}$, $v=\text{const.}$ of the congruence G of lines MM_1 are its developables. We say that the nets N and N_1 are *conjugate* to the congruence G, since the developables of the latter meet the surfaces on which N and N_1 lie in these nets. Two nets so related to a congruence are said to be in the relation of a *transformation* T with one another. Conversely, any two nets so related can be defined analytically by (2) and (3), as we have shown previously.*

In § 2 we show that if two nets N and N_1 are in the relation of a transformation T, the same is true of their first Laplace transforms, and likewise their minus first Laplace transforms. This result holds for nets in space of any order.

Any three of the functions x are the homogeneous coördinates of the planar net, say $P(x)$, into which $N(x)$ is projected from a suitable vertex of the coordinate tetrahedron upon the opposite face. Hence the preceding results can be applied to the transformation of planar nets. The lines of the congruence G project into lines of the plane, and thus the developables of the congruence lose their significance, and these transformations in general are not of much interest. However, there are certain types of planar nets characterized, for example, by geometric properties. The above general method can be used to obtain transformations of nets of a type into nets of the same type.

We consider transformations K of planar nets whose equation (1) has equal invariants into nets of the same kind. In a previous paper† we have shown how one can determine by quadratures a surface S whose asymptotic curves project into any given planar net with equal invariants. We show in § 4 that a transformation K of the planar net is equivalent to the determination of a surface S_1 such that S and S_1 are the focal surfaces of a W-congruence. We make use of these results in § 5 to determine the equations of a W-congruence.

There are certain planar nets with equal invariants which are reproduced after three transformations of Laplace. We refer to them as *nets of period* 3. They are of three types. The remainder of the paper is devoted to a study of the transformations K of these nets into nets of the same kind. A theorem of permutability of these transformations is established. We discover also another transformation of nets of period 3, purely analytic in character, and find that their determination and that of transformations K reduce to equivalent analytical problems.

---

* *Transactions of the American Mathematical Society*, Vol. XVIII (1917), pp. 97–124.

† *Annals of Mathematics*, Series 2, Vol. XVIII (1917), pp. 221–225.

In § 12 we show that the preceding results can be interpreted as giving transformations of certain surfaces discovered by Tzitzeica.

2. **Laplace Transformations and Transformations T.** The tangents to the curves $v=$const. of a net N are tangent to the curves $u=$const. of a net $(N)_1$, and the tangents to the curves $u=$const. of N are tangent to the curves $v=$const. of a net $(N)_{-1}$. The nets $(N)_1$ and $(N)_{-1}$ are called the *first* and *minus first Laplace transforms* of N. Their respective homogeneous coordinates y and z can be given the forms

$$y=\frac{\partial x}{\partial u}-bx, \quad z=\frac{\partial x}{\partial v}-ax.*$$

We shall prove the theorem:

If N_1 is a T transform of N determined by a function θ, the first and minus first Laplace transforms of N_1 are T transforms of the first and minus first Laplace transforms of N by means of the respective functions

$$\frac{\partial\theta}{\partial u}-b\theta, \quad \frac{\partial\theta}{\partial v}-a\theta.$$

In consequence of (4) it follows that the homogeneous coordinates of the first Laplace transform of N_1 are of the form

$$\frac{\partial x_1}{\partial u}+\frac{h}{l}\left(\frac{\partial\log\theta}{\partial u}-b\right)x_1.$$

By means of (2) this is reducible to such a form that we can take for the coordinates y_1 of this transform the expressions

$$y_1=\frac{l\dfrac{\partial}{\partial u}\left(\dfrac{x}{\theta}\right)}{\dfrac{\partial}{\partial u}\log\theta-b}+x_1.$$

Making use of the fact that x and θ are solutions of equation (1), we obtain

$$\frac{\partial y_1}{\partial u}=\bar{h}\frac{\partial}{\partial u}\left(\frac{y}{\bar\theta}\right), \quad \frac{\partial y_1}{\partial v}=\bar{l}\frac{\partial}{\partial v}\left(\frac{y}{\bar\theta}\right),$$

where

$$\bar\theta=\frac{\partial\theta}{\partial u}-b\theta, \quad \bar{h}=\frac{\dfrac{\partial\log\theta}{\partial u}-b}{k}+l,$$

---

*Darboux, *Leçons*, Vol. II, pp. 27, 28.

k being the invariant of equation (1) defined by

$$k=\frac{\partial b}{\partial v}-ab-c.$$

Since the above equations necessarily satisfy the conditions of integrability, we have proved that the two nets are in the relation of a transformation T. By similar means we show that the minus first transforms are so related.

3. **Equations of Transformations T.** The three homogeneous coordinates $x^{(1)}, x^{(2)}, x^{(3)}$ of a planar net $P(x)$ are solutions not only of equation (1), but also of two equations of the form

$$\frac{\partial^2\theta}{\partial u^2}=a'\frac{\partial\theta}{\partial u}+b'\frac{\partial\theta}{\partial v}+c'\theta, \quad \frac{\partial^2\theta}{\partial v^2}=a''\frac{\partial\theta}{\partial u}+b''\frac{\partial\theta}{\partial v}+c''\theta, \tag{6}$$

where the coefficients are functions of u and v, which must be such that the conditions of integrability

$$\frac{\partial}{\partial v}\left(\frac{\partial^2 x}{\partial u^2}\right)=\frac{\partial}{\partial u}\left(\frac{\partial^2 x}{\partial u\partial v}\right), \quad \frac{\partial}{\partial u}\left(\frac{\partial^2 x}{\partial v^2}\right)=\frac{\partial}{\partial v}\left(\frac{\partial^2 x}{\partial u\partial v}\right),$$

are satisfied by all three functions x. This leads to the following six conditions:

$$\left.\begin{aligned}
&\frac{\partial a'}{\partial v}+b'a''=\frac{\partial a}{\partial u}+ab+c, \quad \frac{\partial b'}{\partial v}+a'b+b'b''+c'=\frac{\partial b}{\partial u}+ab'+b^2,\\
&\frac{\partial c'}{\partial v}+a'c+b'c''=\frac{\partial c}{\partial u}+ac'+bc, \quad \frac{\partial b''}{\partial u}+b'a''=\frac{\partial b}{\partial v}+ab+c,\\
&\frac{\partial a''}{\partial u}+a'a''+ab''+c''=\frac{\partial a}{\partial v}+a''b+a^2, \quad \frac{\partial c''}{\partial u}+a''c'+b''c=\frac{\partial c}{\partial v}+ac+bc''.
\end{aligned}\right\} \tag{7}$$

By making use of equations (6), we are able to determine functions k, m, n so that, for all three x's,

$$x_1=kx+m\frac{\partial x}{\partial u}+n\frac{\partial x}{\partial v}. \tag{8}$$

In fact, when this expression is substituted in (2), we find that k, m, n must satisfy the completely integrable system of equations

$$\left.\begin{aligned}
&\frac{\partial k}{\partial u}+mc'+nc+\frac{h}{\theta^2}\frac{\partial\theta}{\partial u}=0, \quad \frac{\partial m}{\partial u}+k+ma'+na-\frac{h}{\theta}=0,\\
&\frac{\partial n}{\partial u}+mb'+nb=0, \quad \frac{\partial k}{\partial v}+mc+nc''+\frac{l}{\theta^2}\frac{\partial\theta}{\partial v}=0,\\
&\frac{\partial m}{\partial v}+ma+na''=0, \quad \frac{\partial n}{\partial v}+k+mb+nb''-\frac{l}{\theta}=0.
\end{aligned}\right\} \tag{9}$$

4. **Planar Nets with Equal Invariants.** When $P(x)$ and $P_1(x_1)$ are the projections of two nets $N(x)$ and $N_1(x_1)$ in the relation of a transformation T, the points of the plane whose corresponding coordinates are given by (5) are points of contact of the line $\overline{PP_1}$ with its envelope as u or v varies. We consider the case when these points are harmonic to P and P_1, and say that P and P_1 are in the relation of a *transformation* K. From (5) it follows that we must have $l=-h$, and consequently from (3) that

$$\frac{\partial a}{\partial u}=\frac{\partial b}{\partial v},$$

that is, equation (1) has equal invariants.* By making a suitable choice of the homogeneous coordinates x in this case we can take equation (1) in the form

$$\frac{\partial^2\theta}{\partial u\partial v}=c\theta. \tag{10}$$

Now, in all generality we have from (3)

$$h=-l=-\theta^2, \tag{11}$$

so that, if x_1 in (2) be replaced by $x_1\theta$, we have

$$\frac{\partial}{\partial u}(x_1\theta)=-\theta\frac{\partial x}{\partial u}+x\frac{\partial\theta}{\partial u}, \quad \frac{\partial}{\partial v}(x_1\theta)=\theta\frac{\partial x}{\partial v}-x\frac{\partial\theta}{\partial v}, \tag{12}$$

and the point equation of N_1 is

$$\frac{\partial^2\theta_1}{\partial u\partial v}=\theta\frac{\partial^2}{\partial u\partial v}\frac{1}{\theta}\cdot\theta_1. \tag{13}$$

We have shown† that if we define three functions $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ by

$$\frac{\partial y^{(1)}}{\partial u}=x^{(2)}\frac{\partial x^{(3)}}{\partial u}-x^{(3)}\frac{\partial x^{(2)}}{\partial u}, \quad \frac{\partial y^{(1)}}{\partial v}=-x^{(2)}\frac{\partial x^{(3)}}{\partial v}+x^{(3)}\frac{\partial x^{(2)}}{\partial v}, \tag{14}$$

and by similar equations obtained by permuting the hyperscripts cyclically, the function $x^{(4)}$ given by

$$x^{(4)}=x^{(1)}y^{(1)}+x^{(2)}y^{(2)}+x^{(3)}y^{(3)}, \tag{15}$$

and $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are the homogeneous point coordinates of a surface S, upon which the parametric curves are the asymptotic lines. These lines and the planar net are in perspective relation from the point (0, 0, 0, 1).

---

* This is in accordance with the similar result for nets in space first proved by Koenig's *Comptes Rendus*, Vol. CXIII (1891), p. 1022.

† *Annals, loc. cit.* The letters x and y must be interchanged to give the equations in use in the present paper.

Since $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are solutions of (10), equations (14) are of the form of equations of Lelieuvre* $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, 1, are the point coordinates of a surface Σ referred to its asymptotic lines. Moreover, from (15) it is seen that Σ and S are polar reciprocal with respect to the quadric

$$Z^{(1)^2}+Z^{(2)^2}+Z^{(3)^2}=Z^{(4)^2}. \tag{16}$$

By means of the three functions $x_1^{(1)}$, $x_1^{(2)}$ $x_1^{(3)}$ obtained from the quadratures (12), we define three functions $y_1^{(1)}$, $y_1^{(2)}$, $y_1^{(3)}$ by

$$\frac{\partial y_1^{(1)}}{\partial u}=x_1^{(2)}\frac{\partial x_1^{(3)}}{\partial u}-x_1^{(3)}\frac{\partial x_1^{(2)}}{\partial u}, \quad \frac{\partial y_1^{(1)}}{\partial v}=-x_1^{(2)}\frac{\partial x_1^{(3)}}{\partial v}+x_1^{(3)}\frac{\partial x_1^{(2)}}{\partial v}, \tag{17}$$

and similar equations obtained by permuting the hyperscripts cyclically. Now, as seen above, the functions $y_1^{(1)}$, $y_1^{(2)}$, $y_1^{(3)}$, 1, are the point coordinates of a surface Σ_1, referred to its asymptotic lines. Moreover, it is readily found† that

$$y_1^{(1)}=y^{(1)}+x^{(2)}x_1^{(3)}-x^{(3)}x_1^{(2)}. \tag{18}$$

From (14) and (15) it follows that the equation of the tangent plane to Σ is

$$x^{(1)}z^{(1)}+x^{(2)}z^{(2)}+x^{(3)}z^{(3)}-x^{(3)}=0,$$

the z's being current coordinates. It is readily seen that this plane passes through the point $(y_1^{(1)}, y_1^{(2)}, y_1^{(3)}, 1)$ of Σ_1. In like manner we show that the tangent plane to Σ_1 passes through the corresponding point on Σ. Hence Σ and Σ_1 are the focal surfaces of the congruence of lines joining corresponding points of these surfaces, which accordingly is a W-congruence.

By means of $y_1^{(1)}$, $y_1^{(2)}$, $y_1^{(3)}$ we obtain, as by (15), the surface S_1 whose asymptotic curves project from $(0, 0, 0, 1)$ into the net $P_1(x_1)$. Since S_1 is the polar transform of Σ_1 with respect to the quadric (16), and this transformation changes a W-congruence into a W-congruence, we have the theorem:

When two planar nets with equal invariants are in the relation of a transformation K, the two surfaces, whose asymptotic lines are the perspectives of the curves of the nets from a point, are the focal surfaces of a W-congruence.

5. **Equations of a W-congruence.** It is well known that the direction-cosines of the normal to a surface referred to its asymptotic lines are solutions of the same equation of Laplace with equal invariants.‡ It is possible to choose a suitable factor so that the direction-parameters (that is, quantities proportional to the direction-cosines) shall satisfy by an equation of the form (10).

* E., p. 193. A reference of this sort is to the author's "Differential Geometry."

† E., p. 418.

‡ E., p. 194.

In this case we say that the parameters are in the *normal form.* From (14) it is seen that $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are the direction-parameters of Σ in the normal form.

From (14) we have, by differentiation and reduction, with the aid of (6),

$$\frac{\partial^2 y^{(1)}}{\partial u^2}=a'\frac{\partial y^{(1)}}{\partial u}-b'\frac{\partial y^{(1)}}{\partial v}, \quad \frac{\partial^2 y^{(1)}}{\partial v^2}=-a''\frac{\partial y^{(1)}}{\partial u}+b''\frac{\partial y^{(1)}}{\partial v}, \tag{19}$$

and

$$\frac{\partial^2 y^{(1)}}{\partial u\partial v}=\frac{\partial \log x^{(3)}}{\partial v}\frac{\partial y^{(1)}}{\partial u}+\frac{\partial \log x^{(3)}}{\partial u}\frac{\partial y^{(1)}}{\partial v}. \tag{20}$$

The coordinates $y^{(2)}$, $y^{(3)}$ also satisfy (19), and $y^{(2)}$ satisfies (20). Hence we have the theorem:

When the asymptotic lines on a surface are projected orthogonally on a plane, taken as a coordinate plane, the equation satisfied by the cartesian coordinates of this planar net is (20), *where* $x^{(3)}$ *is the direction-parameter in the normal form of the normal to the surface with respect to the normal to the plane.*

When two surfaces Σ and Σ_1 are the focal surfaces of a W-congruence, we say that they are in the relation of a *W-transformation.* Equations (18) define the general transformation. They can be given another form by means of equations (8) for the case of transformations K. In accordance with § 3 we replace x_1 and k by $x_1\theta$ and $k\theta$, respectively, and equation (8) becomes

$$x_1=kx+\frac{m}{\theta}\frac{\partial x}{\partial u}+\frac{n}{\theta}\frac{\partial x}{\partial v}. \tag{21}$$

Since now $a=b=0$ and equations (11) hold, equations (9) become

$$\left.\begin{aligned}
&\theta\frac{\partial k}{\partial u}+(k-1)\frac{\partial \theta}{\partial u}+mc'+nc=0, \quad \theta\frac{\partial k}{\partial v}+(k+1)\frac{\partial \theta}{\partial v}+mc+nc''=0,\\
&\frac{\partial m}{\partial u}+k\theta+ma'+\theta=0, \quad \frac{\partial m}{\partial v}+na''=0,\\
&\frac{\partial n}{\partial u}+mb'=0, \quad \frac{\partial n}{\partial v}+k\theta+nb''-\theta=0.
\end{aligned}\right\} \tag{22}$$

Substituting the values from (21) in (18), we get the desired equations of transformation, namely

$$y_1=y+\frac{m}{\theta}\frac{\partial y}{\partial u}-\frac{n}{\theta}\frac{\partial y}{\partial v}. \tag{23}$$

Hence the problem of W-transformations is equivalent to the solution of equation (10) and of the completely integrable system (22).

6. **Nets of Period 3.** We apply the preceding methods to nets for which $a=b=0$, $c'=c''=0$. These nets are characterized by the property that any one of them is reproduced after three transformations of Laplace. In this sense they are of period 3.

It follows from (7) that in the present case

$$a'=\frac{\partial \log c}{\partial u}, \quad b''=\frac{\partial \log c}{\partial v}, \quad b'=\frac{U}{c}, \quad a''=\frac{V}{c},$$

where U and V are arbitrary functions of u and v respectively. One or both of these functions can be taken equal to zero. If they are not zero, by a suitable choice of the parameters u and v of the nets we can take U and V equal to unity. Hence in all generality we have

$$b'=\frac{\varepsilon}{c}, \quad a''=\frac{\eta}{c},$$

where ε and η take the values 0 or 1, and the differential equations of the net are

$$\frac{\partial^2 x}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial x}{\partial u}+\frac{\varepsilon}{c}\frac{\partial x}{\partial v}, \quad \frac{\partial^2 x}{\partial u \partial v}=cx, \quad \frac{\partial^2 x}{\partial v^2}=\frac{\eta}{c}\frac{\partial x}{\partial u}+\frac{\partial \log c}{\partial v}\frac{\partial x}{\partial v}. \tag{24}$$

All the conditions (7) are satisfied provided that c satisfies

$$\frac{\partial^2}{\partial u \partial v}\log c=c-\frac{\varepsilon\eta}{c^2}. \tag{25}$$

For the present case equations (22) reduce to

$$\left.\begin{aligned}
&\theta\frac{\partial k}{\partial u}+nc+(k-1)\frac{\partial \theta}{\partial u}=0, \quad \theta\frac{\partial k}{\partial v}+mc+(k+1)\frac{\partial \theta}{\partial v}=0,\\
&\frac{\partial m}{\partial u}+k\theta+m\frac{\partial \log c}{\partial u}+\theta=0, \quad \frac{\partial m}{\partial v}+\frac{n\eta}{c}=0,\\
&\frac{\partial n}{\partial u}+\frac{m\varepsilon}{c}=0, \quad \frac{\partial n}{\partial v}+k\theta+n\frac{\partial \log c}{\partial v}-\theta=0.
\end{aligned}\right\} \tag{26}$$

Eliminating k from the second and third of these equations, we get

$$\frac{\partial^2 m}{\partial u \partial v}+\frac{\partial \log c}{\partial u}\frac{\partial m}{\partial v}+\left(\frac{\partial^2 \log c}{\partial u \partial v}-c\right)m=0.$$

It is readily seen that $\frac{\partial \theta}{\partial v}/c$ is a solution of this equation, so that we replace m by $\mu\frac{\partial \theta}{\partial v}/c$. Similarly, we replace n by $\nu\frac{\partial \theta}{\partial u}/c$. Then (21) becomes

$$x_1=xk+\frac{\mu}{c\theta}\frac{\partial \theta}{\partial v}\frac{\partial x}{\partial u}+\frac{\nu}{c\theta}\frac{\partial \theta}{\partial u}\frac{\partial x}{\partial v}, \tag{27}$$

and (26) reduces to

$$\left.\begin{aligned}
&\frac{\partial k}{\partial u}+(k+\nu-1)\frac{1}{\theta}\frac{\partial\theta}{\partial u}=0,\quad \frac{\partial k}{\partial v}+(k+\mu+1)\frac{1}{\theta}\frac{\partial\theta}{\partial v}=0,\\
&\frac{\partial\mu}{\partial u}\frac{\partial\theta}{\partial v}+c\theta(k+\mu+1)=0,\quad \frac{\partial\mu}{\partial v}\frac{\partial\theta}{\partial u}+c\theta(k+\nu-1)=0,\\
&\frac{\partial\nu}{\partial u}\frac{\partial\theta}{\partial u}+\nu\frac{\partial^2\theta}{\partial u^2}-\nu\frac{\partial\theta}{\partial u}\frac{\partial\log c}{\partial u}+\frac{\varepsilon\mu}{c}\frac{\partial\theta}{\partial v}=0,\\
&\frac{\partial\mu}{\partial v}\frac{\partial\theta}{\partial v}+\mu\frac{\partial^2\theta}{\partial v^2}-\mu\frac{\partial\theta}{\partial v}\frac{\partial\log c}{\partial v}+\frac{\eta\mu}{c}\frac{\partial\theta}{\partial u}=0.
\end{aligned}\right\}\tag{28}$$

If the net $N_1(x_1)$ is to be of the same kind, we must have

$$\frac{\partial^2x_1}{\partial u^2}=\frac{\partial\log c_1}{\partial u}\frac{\partial x_1}{\partial u}+\frac{U}{c_1}\frac{\partial x_1}{\partial v},\quad \frac{\partial^2x_1}{\partial u\partial v}=c_1x_1,\quad \frac{\partial^2x_1}{\partial v^2}=\frac{V}{c_1}\frac{\partial x_1}{\partial u}+\frac{\partial\log c_1}{\partial v}\frac{\partial x_1}{\partial v},\tag{29}$$

where, as follows from (13), c_1 is given by

$$c_1=\theta\frac{\partial^2}{\partial u\partial v}\frac{1}{\theta}=-c+\frac{2}{\theta^2}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}.\tag{30}$$

When the expression for x_1, as given by (27), is differentiated and substituted in the first and third of (29), the resulting equations are reducible by means of (24) and (28) to equations of the form

$$Ax+B\frac{\partial x}{\partial u}+C\frac{\partial x}{\partial v}=0,\quad A_1x+B_1\frac{\partial x}{\partial u}+C_1\frac{\partial x}{\partial v}=0,$$

where A, B, C, A_1, B_1, C_1 are determinate functions of u and v. As the above equations must be satisfied by $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$, the coefficients must vanish identically. Hence we have the following equations of condition:

$$\left.\begin{aligned}
&U(k+\mu+1)=0,\quad V(k+\nu-1)=0,\\
&U\left(\frac{2\nu}{c\theta^2(1-k)}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}-1\right)+\varepsilon\left(1-\frac{2}{c\theta^2}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}\right)=0,\\
&V\left(\frac{2\mu}{c\theta^2(1+k)}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}+1\right)-\eta\left(1-\frac{2}{c\theta^2}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}\right)=0,\\
&\frac{\partial^2\theta}{\partial u^2}=\frac{\partial\log c}{\partial u}\frac{\partial\theta}{\partial u}+\frac{1+k}{1-k}\frac{U}{c}\frac{\partial\theta}{\partial v},\quad \frac{\partial^2\theta}{\partial v^2}=\frac{1-k}{1+k}\frac{V}{c}\frac{\partial\theta}{\partial u}+\frac{\partial\log c}{\partial v}\frac{\partial\theta}{\partial v}.
\end{aligned}\right\}\tag{31}$$

We break up the discussion of the problem into three cases.

7. **When $U\neq0$, $V\neq0$. Nets of the First Type.** Comparing equations (28) and (31), we find that k, μ and ν are constants such that

$$k+\mu+1=0,\quad k+\nu-1=0.\tag{32}$$

Moreover, we have

$$U=\varepsilon=1, \quad V=\eta=1.$$

If k and x_1 be replaced by $1/e$ and x_1/e, equation (27) becomes

$$x_1=x-\frac{1+e}{c\theta}\frac{\partial\theta}{\partial v}\frac{\partial x}{\partial u}-\frac{1-e}{c\theta}\frac{\partial\theta}{\partial u}\frac{\partial x}{\partial v}. \tag{33}$$

The equations of condition (28) and (31) are satisfied, provided θ is a solution of the system

$$\left.\begin{aligned}&\frac{\partial^2\theta}{\partial u^2}=\frac{\partial\log c}{\partial u}\frac{\partial\theta}{\partial u}+\frac{e+1}{e-1}\frac{\varepsilon}{c}\frac{\partial\theta}{\partial v}, \quad \frac{\partial^2\theta}{\partial u\partial v}=c\theta,\\ &\frac{\partial^2\theta}{\partial v^2}=\frac{e-1}{e+1}\frac{\eta}{c}\frac{\partial\theta}{\partial u}+\frac{\partial\log c}{\partial v}\frac{\partial\theta}{\partial v},\end{aligned}\right\} \tag{34}$$

with $\varepsilon=\eta=1$. We say that a net with equations (24) where $\varepsilon=\eta=1$ is of the *first type.*

The general solution of equations (34) involves three arbitrary constants in addition to e. Hence we have the theorem:

A planar net of period 3 of the first type admits ∞^4 transforms of the same kind.

We remark that equations (29) for N_1 are of the same form as (24).

From the results of § 2 it follows that the other two nets forming with N a closed cycle under Laplace transformations are in the relation of transformations K with the two nets forming with N_1 a closed cycle. This remark applies equally well to nets of the second and third types to be discussed in the following sections.

8. **When $U=0$, $V\neq 0$. Nets of the Second Type.** From the second of (31) it follows that $\varepsilon=0$ or $c_1=0$. Assuming $\varepsilon=0$, we find, as in the previous case, that k, μ and ν are constants in the relations (32); also that $V=\eta=1$. Now (25) becomes

$$\frac{\partial^2\log c}{\partial u\partial v}=c, \tag{35}$$

of which the general integral is

$$c=-\frac{2U'V'}{(1+UV)^2}, \tag{36}$$

where U and V are arbitrary functions of u and v respectively, and the primes indicate differentiation with respect to the argument. When $\varepsilon=0$, $\eta=1$, the net is of the *second type.*

When the value (36) of c is substituted in the second of (34), we find the general integral of the resulting equation; it is

$$\theta=\frac{U_1'}{U'}+\frac{V_1'}{V'}-2\frac{U_1V+V_1U}{1+UV'}, \tag{37}$$

where U_1 and V_1 are arbitrary functions of u and v respectively.

The fifth of (31) is $\frac{\partial^2\theta}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial\theta}{\partial u}$, of which the general integral is

$$\frac{\partial\theta}{\partial u}=c\phi(v),$$

ϕ being an arbitrary function of v. When the value (37) of θ is substituted in this equation, it becomes

$$\frac{1}{U'}\left(\frac{U_1'}{U'}\right)'(1+UV)^2-2V\frac{U_1'}{U'}(1+UV)+2(U_1V^2-V_1)+2\phi(v)V'=0. \tag{38}$$

Differentiating with respect to u, we get

$$\left[\frac{1}{U'}\left(\frac{U_1'}{U'}\right)'\right]'(1+UV)^2=0.$$

Hence

$$U_1=aU^2+bU+c, \tag{39}$$

where a, b, c are constants. Substituting in (38), we find

$$a-bV+cV^2=V_1-\phi V'.$$

If we replace $\phi V'$ by V_2, so that

$$V_1=a-bV+cV^2+V_2, \tag{40}$$

equation (37) reduces to

$$\theta=\frac{V_2'}{V'}-\frac{2UV_2}{1+UV}. \tag{41}$$

The last of (31) remains to be satisfied; it may be taken in the form of the last of (34). Substituting the expression (41) for θ in this equation, we have for the determination of V_2 the equation

$$V'\left(\frac{V_2'}{V'}\right)''-V''\left(\frac{V_2'}{V'}\right)'-\frac{e-1}{e+1}V_2=0. \tag{42}$$

Thus the solution of this equation determines the transformations of nets of the second type into nets of the same kind. As before we have the theorem:

A planar net of the second type admits ∞^4 transforms of the same kind, and their determination requires the integration of an ordinary differential equation of the third order.

It remains for us to consider the second possibility for the satisfaction of the second of (31), namely, $\varepsilon=1$, $c_1=0$. Then, from (31) we get the two equa-

tions (32), and from the first two of (28) it follows that k, μ and ν are constants. Then the fifth equations of (28) and (31) are inconsistent.

From the results of §§ 6, 7 it follows that a net of the first or second type does not admit a transformation K into a net of the second or first type respectively.

So far as the actual determination of nets of the second type goes, it will be shown that it is a problem of the integration of an ordinary differential equation of the third order involving an arbitrary function. In fact, when in (1) and (6)

$$a=b=b'=c'=c''=0,$$

equations (7) are equivalent to

$$a'=\frac{\partial \log c}{\partial u}, \quad b''=\frac{\partial \log c}{\partial v}, \quad a''=\frac{2V_1}{c}, \quad \frac{\partial^2 \log c}{\partial u \partial v}=c,$$

where V_1 is an arbitrary function of v alone. The function c is given by (36). If U and V be taken as the parameters and be replaced by u and v, in place of (24) we have the system

$$\frac{\partial^2 x}{\partial u^2}=\frac{-2v}{1+uv}\frac{\partial x}{\partial u}, \quad \frac{\partial^2 x}{\partial u \partial v}=\frac{-2}{(1+uv)^2}x, \quad \frac{\partial^2 x}{\partial v^2}=V_1(1+uv)^2\frac{\partial x}{\partial u}-\frac{2u}{1+uv}\frac{\partial x}{\partial v}.$$

The integral of the second of these equations is of the form (37) with U and V replaced by u and v. In order that the other two equations shall be satisfied, we find, similarly to (41), that the coordinates of the net are of the form

$$x^{(i)}=\psi_i(v)-\frac{2v\psi_i}{1+uv},$$

where the functions ψ_i are linearly independent solutions of the equation

$$\psi'''-2V_1\psi=0.$$

Thus the above statement has been proved.

9. **Transformations L of Nets of the First and Second Types.** The equation (25) is such that if $c(u, v)$ is a solution, so also is $c_m \equiv c(mu, v/m)$, where m is any constant. Hence with a net N of the first or second type which has equations of the form (24) in which c is known there is associated a net N_m whose equations are

$$\frac{\partial^2 x_m}{\partial u^2}=\frac{\partial}{\partial u}\log c_m\frac{\partial x_m}{\partial u}+\frac{\varepsilon}{c_m}\frac{\partial x_m}{\partial v}, \quad \frac{\partial^2 x_m}{\partial u \partial v}=c_m x_m, \quad \frac{\partial^2 x_m}{\partial v^2}=\frac{\eta}{c_m}\frac{\partial x_m}{\partial u}+\frac{\partial}{\partial v}\log c_m\frac{\partial x_m}{\partial v}.$$

As this transformation from N to N_m is suggestive of the so-called Lie transformation of pseudo-spherical surfaces,* we call it a *transformation* L_m.

* E., p. 280.

We return to the consideration of equations (34) and note that if we effect the change of variables given by

$$\mu=\sqrt[3]{\frac{e-1}{e+1}}u_1, \quad v=\sqrt[3]{\frac{e+1}{e-1}}v_1,$$

these equations reduce to

$$\frac{\partial^2\theta}{\partial u_1^2}=\frac{\partial}{\partial u_1}\log c\frac{\partial\theta}{\partial u_1}+\frac{\varepsilon}{c}\frac{\partial\theta}{\partial v_1}, \quad \frac{\partial^2\theta}{\partial u_1\partial v_1}=c\theta, \quad \frac{\partial^2\theta}{\partial v_1^2}=\frac{\eta}{c}\frac{\partial\theta}{\partial u_1}+\frac{\partial}{\partial v_1}\log c\frac{\partial\theta}{\partial v_1}.$$

Comparing these equations with the preceding set, we see that the general solution of these equations, and, consequently, of (34), is given by

$$\theta=\sum_1^3 a_i x_m^{(i)}\left(\frac{u}{m}, mv\right),$$

where a_i are constants, $x_m^{(i)}(u, v)$ are the coordinates of the net N_m and

$$m=\sqrt[3]{\frac{e-1}{e+1}}.$$

Hence we have the theorem:

The complete determination of transformations K of nets of the first and second types is the same analytical problem as the complete determination of transformations L.

10. **When $U=V=0$. Nets of the Third Type.** We consider first the case where $c_1\neq 0$. Then, as follows from (31), $\varepsilon=\eta=0$. In this case the net N is said to be of the *third type.* Now the last two of (31) reduce to (34) with $\varepsilon=\eta=0$, and c is given by (36). Hence θ is of the form (37) with U_1 given by (39) and V_1 by a similar expression, say

$$V_1=a_1V^2+b_1V+c_1.$$

Accordingly θ is reducible to the form

$$\theta=[f(1-UV)+gU+hV]/(1+UV),$$

where f, g, h are arbitrary constants whose form in terms of the constants $a_1, b_1, \ldots, c_1$ is unessential.

When $\varepsilon=\eta=0$, it is not necessary to make a special choice of parameters u and v so as to reduce the equations of the net to the form (24). As the parameters are consequently undetermined, we can assume that they are chosen so that the most general form of θ is

$$\theta=[f(1-uv)+gu+hv]/(1+uv). \tag{43}$$

From the last two of (28) we have

$$\mu=U_1, \quad \nu=V_1,$$

where U_1 and V_1 are functions of u and v respectively. The third and fourth of (28) reduce to

$$\theta(k+\mu+1)=\frac{U_1'}{2}(h-2fu-gu^2),\quad \theta(k+\nu-1)=\frac{V_1'}{2}(g-2fv-hv^2), \tag{44}$$

expressions which are consistent with the first two of (28).

If equations (44) be subtracted from one another, we get

$$\begin{aligned}&[f(1-uv)+gu+hv](U_1-V_1+2)\\ &\qquad=\left[\frac{U_1'}{2}(h-2fu-gu^2)-\frac{V_1'}{2}(g-2fv-hv^2)\right](1+uv).\end{aligned} \tag{45}$$

Differentiating with respect to u and v, we have

$$\begin{aligned}&[(h-fu)U_1]'-[(h-gv)V_1]'-2f\\ &\qquad=\left[\frac{uU_1'}{2}(h-2fu+gu^2)\right]'-\left[\frac{vV_1'}{2}(g-2fv+hv^2)\right]'.\end{aligned}$$

This equation may be replaced by the two

$$\left.\begin{aligned}\frac{U_1'}{2}(h-2fu+gu^2)&=\left(\frac{h}{u}-f\right)U_1-2f+\alpha+\frac{\beta}{u},\\ \frac{V_1'}{2}(g-2fv+hv^2)&=\left(\frac{g}{v}-f\right)V_1+\alpha+\frac{\gamma}{v},\end{aligned}\right\} \tag{46}$$

where α, β, γ are constants. Substituting in (45), we obtain

$$\left(2f+gu-\frac{h}{u}\right)U_1-\left(2f+hv-\frac{g}{v}\right)V_1+4f+2gu+2hv-(1+uv)\left(\frac{\beta}{u}-\frac{\gamma}{v}\right)=0.$$

This equation is equivalent to

$$\begin{aligned}(h-2fu-gu^2)U_1&=(4f+\delta)u+2gu^2-\beta+\gamma u^2,\\ (g-2fv-hv^2)V_1&=\delta v-2hv^2-\gamma+\beta v^2,\end{aligned}$$

where δ is a constant. These results are consistent with (46) when $\delta=-2\alpha$, in which case we have from (44) for the determination of k,

$$k[f(1-uv)+gu+hv]=(f-\alpha)(1-uv)+(g+\gamma)u+(\beta-h)v.$$

When the constants α, β, γ satisfy the conditions

$$(\beta/h)-2=\gamma/g=-\alpha/f,$$

and only in this case, U_1 and V_1 are constants. Then k is constant, and the equations of the transformation are of the form (33). As in the other two cases, we have the theorem:

A planar net of the third type admits ∞^4 transforms of the same kind, and they can be found without quadrature.

We consider now the case where $c_1=0$. From (30) it follows that

$$\theta=1/(U+V),$$

where U and V are functions of u and v respectively. When this value is substituted in the second of (34), we find

$$c=2U'V'/(U+V)^2.$$

In order that this value may satisfy (25), we must have $\varepsilon\eta=0$. The last two of equations (31) reduce to

$$\frac{\partial^2\theta}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial\theta}{\partial u}, \quad \frac{\partial^2\theta}{\partial v^2}=\frac{\partial \log c}{\partial v}\frac{\partial\theta}{\partial v},$$

which are satisfied identically by the above values, whatever be U and V.

From the fifth and fourth of (28) we get

$$\nu=V_1, \quad k=(U+V)(V_1'/2V')-V_1'+1,$$

where V_1 is an arbitrary function of v. From the second of (28) we have

$$\mu=\frac{(U+V)^2}{V'}\left(\frac{V_1'}{2V'}\right)'-\frac{V_1'}{V'}(U+V)+V_1-2.$$

These values satisfy the first and third of equations (28) identically, but in order that the last of (28) be satisfied, V_1 must be such that

$$\left(\frac{V_1'}{V'}\right)''-\frac{V''}{V'}\left(\frac{V_1'}{V'}\right)'+\eta\frac{V_1}{V'}=0.$$

The function k is constant only when V_1 is constant. Then from the above equation $\eta=0$.

The results just obtained show that nets of the second and third types admit transformation K into nets whose equations are given by (1) and (6) when

$$a=b=c=b'=c'=a''=c''=0.$$

11. **Theorem of Permutability.** In this section we establish a theorem of permutability of the transformations of nets of period 3.

We take two functions θ_i and two constants e_i $(i=1, 2)$ satisfying the systems analogous to (34), namely

$$\left.\begin{aligned}&\frac{\partial^2\theta_i}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial\theta_i}{\partial u}+\frac{e_i+1}{e_i-1}\frac{\varepsilon_i}{c}\frac{\partial\theta_i}{\partial v}, \quad \frac{\partial^2\theta_i}{\partial u\partial v}=c\theta_i,\\ &\frac{\partial^2\theta_i}{\partial v^2}=\frac{e_i-1}{e_i+1}\frac{\eta_i}{c}\frac{\partial\theta_i}{\partial u}+\frac{\partial \log c}{\partial v}\frac{\partial\theta_i}{\partial v}.\end{aligned}\qquad (i=1, 2)\right\} \quad (47)$$

Thus far we suppose ε_i and η_i capable of assuming either values 0 or 1.

By means of θ_1 and θ_2 we effect transformations of N into N_1 and N_2 respectively. From the form of (12) it follows that there exists functions θ_{12} and θ_{21} defined by

$$\frac{\partial}{\partial u}(\theta_i\theta_{ij})=-\theta_i\frac{\partial\theta_j}{\partial u}+\theta_j\frac{\partial\theta_i}{\partial u},\quad \frac{\partial}{\partial v}(\theta_i\theta_{ij})=\theta_i\frac{\partial\theta_j}{\partial v}-\theta_j\frac{\partial\theta_i}{\partial v}.\quad \left(\begin{matrix}i=1,2,\\ j=1,2,\end{matrix}\ i\neq j\right) \tag{48}$$

Evidently the constants of integration in these equations can be chosen so that

$$\theta_1\theta_{12}+\theta_2\theta_{21}=0.$$

Hereafter we assume that the functions θ_{12} and θ_{21} are paired in this way, and there are ∞^1 such pairs.

The functions θ_{12} and θ_{21} can be used to effect transformations T of N_1 and N_2 respectively into the same net N_{12}.* We seek the conditions that N_{12} shall be a net of period 3.

From the preceding investigation it follows that θ_{12} must satisfy the equations

$$\frac{\partial^2\theta_{12}}{\partial u^2}=\frac{\partial\log c_1}{\partial u}\frac{\partial\theta_{12}}{\partial u}+\frac{e_{12}+1}{e_{12}-1}\frac{\varepsilon_{12}}{c_1}\frac{\partial\theta_{12}}{\partial v},\quad \frac{\partial^2\theta_{12}}{\partial v^2}=\frac{e_{12}-1}{e_{12}+1}\frac{\eta_{12}}{c_1}\frac{\partial\theta_{12}}{\partial u}+\frac{\partial\log c_1}{\partial v}\frac{\partial\theta_{12}}{\partial v}, \tag{49}$$

where c_1 is given by (30), and ε_{12} and η_{12} are 0 or 1, as the case may be.

If equations (48) be differentiated and the expressions for $\frac{\partial^2\theta_{12}}{\partial u^2}$ and $\frac{\partial^2\theta_{12}}{\partial v^2}$ be substituted in (49), the resulting equations are reducible to

$$\left.\begin{aligned}
&\theta_{12}\left(\frac{e_1+1}{e_1-1}\varepsilon_1+\frac{e_{12}+1}{e_{12}-1}\varepsilon_{12}\right)+\frac{2\varepsilon_1}{c\theta_1}\frac{e_1+1}{e_1-1}\frac{\partial\theta_1}{\partial v}\frac{\partial\theta_2}{\partial u}-\frac{2\varepsilon_2}{c\theta_1}\frac{e_2+1}{e_2-1}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}\\
&\quad+\theta_2\left(\frac{e_{12}+1}{e_{12}-1}\varepsilon_{12}-\frac{e_1+1}{e_1-1}\varepsilon_1\right)+\theta_1\left(\frac{e_2+1}{e_2-1}\varepsilon_2-\frac{e_{12}+1}{e_{12}-1}\varepsilon_{12}\right)\frac{\partial\theta_2}{\partial v}\Big/\frac{\partial\theta_1}{\partial v}=0,\\
&\theta_{12}\left(\frac{e_1-1}{e_1+1}\eta_1+\frac{e_{12}-1}{e_{12}+1}\eta_{12}\right)+\frac{2\eta_2}{c\theta_1}\frac{e_2-1}{e_2+1}\frac{\partial\theta_1}{\partial v}\frac{\partial\theta_2}{\partial u}-\frac{2\eta_1}{c\theta_1}\frac{e_1-1}{e_1+1}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}\\
&\quad+\theta_2\left(\frac{e_1-1}{e_1+1}\eta_1-\frac{e_{12}-1}{e_{12}+1}\eta_{12}\right)+\theta_1\left(\frac{e_{12}-1}{e_{12}+1}\eta_{12}-\frac{e_2-1}{e_2+1}\eta_2\right)\frac{\partial\theta_2}{\partial u}\Big/\frac{\partial\theta_1}{\partial u}=0.
\end{aligned}\right\} \tag{50}$$

We consider first the case when N is of the first type. As we have seen in § 7, the nets N_1, N_2 and N_{12} are then of the first type. Thus all the ε's and η's are equal to 1. Making this substitution in (50), and eliminating θ_{12} from the resulting equations, we get

$$(e_{12}-e_2)\left(\frac{e_1+1}{e_2+1}\frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v}-\frac{e_1-1}{e_2-1}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}\right)c_1=0.$$

---

* *Transactions, loc. cit.*, §§ 5, 11.

It is readily shown that the expression in the parenthesis vanishes only when $\theta_2/\theta_1=$const., but in this case N_1 and N_2 are the same net. Since $c_1\neq 0$, we have $e_{12}=e_2$. Then equations (50) reduce to the single one

$$(e_1e_2-1)\theta_1\theta_{12}+\frac{(e_1+1)(e_2-1)}{c}\frac{\partial\theta_1}{\partial v}\frac{\partial\theta_2}{\partial u}-\frac{(e_2+1)(e_1-1)}{c}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}+\theta_1\theta_2(e_1-e_2)=0. \quad (51)$$

This value satisfies equations (48). Hence we have the theorem:

If N_1 and N_2 are nets of the first type, which are transforms of a net N of this type, there exists a unique net N_{12} of the first type, which is a transform of N_1 and N_2; and it can be found without quadratures.

From the equations of the theorem of permutability of transformations T * we find that for the present choice of coordinates the coordinates of N_{12} are given by

$$\theta_{12}(x_{12}-x)=\theta_2(x_2-x_1). \quad (52)$$

We suppose now that the nets N, N_1 and N_2 are of the second type. If N_{12} also is to be of this type, the ε's must have the value zero, and the η's one. Then the first of (50) is satisfied identically, and if the second be differentiated with respect to u, and use is made of (48), the result is reducible to

$$\frac{\partial\theta_1}{\partial u}\frac{e_2-e_{12}}{(e_2+1)(e_{12}+1)}=0.$$

Hence $e_{12}=e_2$ and the second of (50) reduces to (51). As this value satisfies the second of (48), we have the theorem:

If N_1 and N_2, nets of the second type, are transforms of a net N of the second type, there exists a unique net N_{12} of the second type which is their transform.

We consider finally the possibility of all four nets N, N_1, N_2, N_{12} being of the third type. In this case equations (50) are satisfied identically, and consequently all the nets N_{12} are of the third type. Let the functions θ_1 and θ_2 determining the transforms N_1 and N_2 be, according to (43),

$$\theta_i=[f_i(1-uv)+g_iu+h_iv](1+uv) \qquad (i=1,2).$$

By differentiation we have

$$\frac{\partial\theta_i}{\partial u}=\frac{-2f_iv+g_i-h_iv^2}{(1+uv)^2}, \quad \frac{\partial\theta_i}{\partial v}=\frac{-2f_iu+h_i-g_iu^2}{(1+uv)^2}.$$

---

* *Transactions*, p. 107.

Substituting these values in (48) and making use of the abbreviation $(ab) \equiv a_1b_2 - a_2b_1$, we get

$$\frac{\partial}{\partial u}(\theta_1\theta_{12}) = \frac{-(fg)-(fh)v^2+(gh)v}{(1+uv)^2}, \quad \frac{\partial}{\partial v}(\theta_1\theta_{12}) = \frac{(fh)+(fg)u^2+(gh)u}{(1+uv)^2}.$$

The integral of these equations is

$$\theta_1\theta_{12} = \frac{-\frac{1}{2}(gh)(1-uv)-(fg)u+(fh)v}{1+uv} + \text{const}.$$

Hence we have the theorem:

If N is a net of the third type, and N_1 and N_2 are two transforms of this type, there can be found without quadrature ∞^1 nets N_{12} which are transforms of N_1 and N_2.

12. **Surfaces of Tzitzeica.** When the homogeneous point coordinates of a net of period 3 are chosen so that they satisfy equations (24), they are the non-homogeneous coordinates of a surface referred to its asymptotic lines. In case $\varepsilon=\eta=0$, the surface is a central quadric. When $\eta=1$, the surface is ruled or not according as $\varepsilon=0$ or 1. These surfaces were discovered by Tzitzeica* in his search for surfaces whose total curvature at each point is in constant ratio with the fourth power of the distance of the tangent plane at the point from a point fixed in space, and constitute the complete solution of the problem.

We have seen in § 8 that the complete determination of the ruled surfaces of Tzitzeica requires the integration of an equation of the third order. Making use of this result and the expressions for the coordinates as there given, Tzitzeica showed† that these ruled surfaces are characterized by the property that their flexnode curve is at infinity.

Since the transformation of the planar nets as given by (33) is reciprocal in character, it follows that the equations of the inverse transformation are of the same form. Interpreted for the surface whose non-homogeneous coordinates are $x^{(1)}, x^{(2)}, x^{(3)}$, we have transformations of these surfaces into surfaces of the same kind, such that a surface and a transform are the focal surfaces of a W-congruence. Tzitzeica‡ announced, without proof, the existence of these transformations.

PRINCETON UNIVERSITY.

---

* *Comptes Rendus*, Vol. CXLIV (1907), p. 1257; Vol. CXLV (1907), p. 1132. Also *Rendiconti di Palermo*, Vol. XXV (1908), pp. 180–189.

† *Comptes Rendus*, Vol. CXLV, *loc. cit.*

‡ *Comptes Rendus*, Vol. CL (1910), pp. 955, 1227.

*Orthogonal Function Sets Arising from Integral Equations.**

BY O. D. KELLOGG.

1. *Introduction.*

As is well known,† $K(x, y)$ being a real, continuous, symmetric function, not identically zero, on the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, there is at least one value of λ for which the integral equation

$$\phi(x) = \lambda \int_0^1 \phi(y) K(x, y)\, dy \tag{1}$$

has a solution, $\phi(x)$, continuous, and not identically zero. Unless $K(x, y)$ is the sum of a finite number of products like $\pm\phi(x)\phi(y)$, there is an infinite number of such values of λ, and, in case it converges uniformly, the following development holds for $K(x, y)$:

$$K(x, y) = \frac{\phi_0(x)\phi_0(y)}{\lambda_0} + \frac{\phi_1(x)\phi_1(y)}{\lambda_1} + \frac{\phi_2(x)\phi_2(y)}{\lambda_2} + \ldots, \tag{2}$$

where $\phi_i(x)$ is the normed solution of (1) corresponding to λ_i. The functions $\phi_0(x), \phi_1(x), \phi_2(x), \ldots\ldots$, form an orthogonal set on the interval $(0, 1)$. They will be called the "harmonics" of the kernel $K(x, y)$, and the corresponding values of λ_i, their "frequencies." The "iterated kernels" are defined by

$$K_j(x, y) = \int_0^1 K_{j-1}(x, r) K(r, y)\, dr, \text{ and } K_0(x, y) = K(x, y). \tag{3}$$

For these, the developments, known to be uniformly convergent ‡ for $i \geqq 2$, hold:

$$K_j(x, y) = \frac{\phi_0(x)\phi_0(y)}{\lambda_0^{j+i}} + \frac{\phi_1(x)\phi_1(y)}{\lambda_1^{j+i}} + \frac{\phi_2(x)\phi_2(y)}{\lambda_2^{j+i}} + \ldots. \tag{4}$$

* Presented to the American Mathematical Society, September 4, 1917.

† See, for instance, Schmidt, "Entwickelung wilkürlicher Functionen nach Systemen vorgeschriebener," Diss. Göttingen, 1915; Bôcher, "An Introduction to the Study of Integral Equations," Cambridge tracts in math. and math. phys., 1909; Kowalewski, "Einführung in die Determinantentheorie," Veit u. Comp., Leipzig, 1909.

‡ Kowalewski, *loc. cit.*, p. 533.

In a previous paper* I have shown that a number of the oscillation properties of the more common orthogonal sets are consequences of continuity, orthogonality, and the added condition:

(D): $D_n(x_0, x_1, \ldots, x_n) > 0$ on the range R: $0 < x_0 < x_1 \ldots x_n < 1$, for $n = 0, 1, 2, \ldots$, where

$$D_n(x_0, x_1, \ldots, x_n) = \begin{vmatrix} \phi_0(x_0), \phi_1(x_0), \ldots, \phi_n(x_0), \\ \phi_0(x_1), \phi_1(x_1), \ldots, \phi_n(x_1), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \phi_0(x_n), \phi_1(x_n), \ldots, \phi_n(x_n), \end{vmatrix}$$

and $D_0(x_0) = \phi_0(x_0)$.

It was stated that it appeared desirable to connect this condition with the theory of integral equations. The following is intended as a contribution to this point.

2. *The Condition on $K(x, y)$.*

To find the necessary and sufficient condition on $K(x, y)$ in order that its harmonics may have the property (D) presents difficulties, since the same harmonics may arise from a variety of kernels, the frequencies being altered. However, a condition may be found which is simple, and which appears to be satisfied in the more common cases. We proceed to indicate considerations which suggest this condition, and which make it appear useful.

It is a property of the more common sets that if $\Phi_{0n}(x) = c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x)$ is a linear combination of the harmonics of $K(x, y)$, the function $\int_0^1 \Phi_{0n}(y) K(x, y)\, dy$ has no more sign changes in the interior of $(0, 1)$ than $\Phi_{0n}(x)$, and in many cases, this property holds for any continuous function $f(x)$. Assuming that it does, let $y_0, y_1, \ldots, y_n$ be any $n+1$ points on R. Choose for $f(x)$ any continuous function $f(x, \varepsilon)$ equal to 0 except on the sub-intervals $(y_i - \varepsilon, y_i + \varepsilon)$, where it is determined so that

$$\int_{y_i-\epsilon}^{y_i+\epsilon} f(x, \varepsilon)\, dx = (-1)^{i-1} c_i,$$

the c_i being independent of ε, and $f(x, \varepsilon)$ not changing sign on any sub-interval. If the c_i are all of one sign, $f(x, \varepsilon)$ will have n sign changes, otherwise fewer.

---

* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII, No. 1 (1916), p. 1. The condition (D) employed in that paper, and in this, which requires determinants of the functions $\phi_i(x)$ to be positive, is in reality no more restrictive than that they be different from zero, as is evident when we reflect that the function $\phi_n(x)$ may be replaced, if necessary, by its negative, without affecting any other hypothesis. But it is more convenient to retain it in the original form.

If now the law of the mean be applied to the integral $\int_0^1 f(y, \varepsilon) K(x, y)\,dy$, and the limit taken as $\varepsilon=0$, the result is

$$f_1(x)=c_0K(x, y_0)-c_1K(x, y_1)+\dots+(-1)^{n-1}c_nK(x, y_n),$$

and it may be shown, on our assumption, that this has no more sign changes that $f(x, \varepsilon)$. In other words, there is no set of positive numbers $a_0, a_1, \dots, a_{n+1}$, and no set of points $x_0, x_1, \dots, x_{n+1}$ on R for which the following equations are possible:

$$\left.\begin{array}{r} c_0K(x_0, y_0)-c_1K(x_0, y_1)+\dots+(-1)^n c_nK(x_0, y_n)=a_0, \\ -[c_0K(x_1, y_0)-c_1K(x_1, y_1)+\dots+(-1)^n c_nK(x_1, y_n)=a_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ (-1)^{n+1}[c_0K(x_{n+1}, y_0)-c_1K(x_{n+1}, y_1)+\dots+(-1)^n c_nK(x_{n+1}, y_n)=a_{n+1}. \end{array}\right\} \quad (5)$$

Now let $M_0, M_1, \dots, M_{n+1}$ be the determinants formed from the matrix

$$\|K(x_i, y_j)\|, \quad i=0, 1, \dots, n+1; \quad j=0, 1, \dots, n \qquad (6)$$

by omitting the first, second, third rows, and so forth. No two of the quantities M_i can differ in sign, since in the contrary case the equation $M_0a_0+M_1a_1+\dots+M_{n+1}a_{n+1}=0$ would be possible with all a_i positive, and for such a_i the equations (5) would be compatible. So we conclude that no two of the determinants

$$K\begin{pmatrix} x_0, x_1, \dots, x_n \\ y_0, y_1, \dots, y_n \end{pmatrix}=\begin{vmatrix} K(x_0, y_0), & K(x_0, y_1), & \dots, & K(x_0, y_n) \\ K(x_1, y_0), & K(x_1, y_1), & \dots, & K(x_1, y_n) \\ \dots & \dots & \dots & \dots \\ K(x_n, y_0), & K(x_n, y_1), & \dots, & K(x_n, y_n) \end{vmatrix} \qquad (7)$$

taken from the same matrix (6), can have opposite signs. If, now, we can pass through a chain of such determinants, different from 0, from any given one to any given second, we may conclude that the determinant (7) is either never negative, or never positive on the region RR: $0<x_0<x_1\dots<x_n<1$. $0<y_0<y_1\dots<y_n<1$. Thus is indicated the condition of the next paragraph,

3. *A Sufficient Condition Upon $K(x, y)$.*

We shall suppose the function $K(x, y)$, in addition to being real, continuous, and symmetric on the square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, to have the property:

(K) : $K\begin{pmatrix} x_0, x_1, \dots, x_n \\ y_0, y_1, \dots, y_n \end{pmatrix}\geqq 0$ on RR for $n=0, 1, 2, \dots$, the *inequality* holding when $x_i=y_i$ for all i. By $K\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is understood $K(x_0, y_0)$.

If this property is not enjoyed by $K(x, y)$, it may be by one of the iterated kernels (3), which has the same harmonics. We proceed to prove the theorem:

If $K(x, y)$ *is real, continuous, and symmetric on the square* $0 \leq x \leq 1$, $0 \leq y \leq 1$, *and has the property* (K), *its harmonics have the property* (D), *with, of course, all the properties derived in the paper referred to.*

We begin with the following special case as a lemma:

If $K(x, y)$ *is real, continuous and symmetric and satisfies* (K) *for* $n=0$, *it has but one frequency of least absolute value, and but one harmonic corresponding to this frequency. This harmonic never vanishes on the interior of the interval* $(0, 1)$, *and its frequency is positive.*

It should be noticed that the property (K) for $n=0$ is retained in the iterated kernels. We start from the first iterated kernel, $K_1(x, y)$, whose frequencies, being the squares of those of $K(x, y)$ are all positive. Its least frequency is thus λ_0^2, which we may assume to be 1 without impairing the argument, since we may replace $K_1(x, y)$ by $\lambda_0^2 K_1(x, y)$, the latter kernel having 1 as least frequency. If more than one harmonic corresponded to 1, the development (2) would take the form:

$$K_1(x, y) = \phi_0(x)\phi_0(y) + \phi_1(x)\phi_1(y) + \dots + \phi_n(x)\phi_n(y) + \frac{\phi_{n+1}(x)\phi_{n+1}(y)}{\lambda_{n+1}^2} + \frac{\phi_{n+2}(x)\phi_{n+2}(y)}{\lambda_{n+2}^2} + \dots, \quad (8)$$

where in the first line appear the harmonics corresponding to 1, and in the second, those corresponding to frequencies greater than 1. Repeated iteration diminishes these later terms, and in the limit we have a function

$$F(x, y) = \phi_0(x)\phi_0(y) + \phi_1(x)\phi_1(y) + \dots + \phi_n(x)\phi_n(y), \quad (9)$$

which is continuous, never negative, and symmetric. The harmonics composing it may be assumed to be normed and orthogonal. $F(x, y)$ may, moreover, be shown to be *positive throughout the whole interior of the square* $0<x<1$, $0<y<1$, as follows. It has positive elements, since it is not identically 0, and it is, for any fixed y, a solution of the integral equation $f(x) = \int_0^1 f(y) K_1(x, y)\, dy$. Let b be a value of y such that $f(x) = F(x, b)$ is positive at some point $0<x<1$. Then $f(x)$ can not be 0 at any interior point, for if it were, there would be an interior point, a, terminating an interval on which $f(x)>0$, and this would lead to the contradiction

$$f(a) = 0 = \int_0^1 f(y) K_1(a, y)\, dy,$$

the integrand being positive near $y=a$, and never negative. Thus $F(x, y)>0$ at all points of the line $y=b$, and being symmetric, at all points of $x=b$. Hence, the argument being repeated with any other value of y, the conclusion is established. We shall now show that $F(x, y)$ being >0 within the square, the sum (9) reduces to a single product.

We must distinguish three cases according to the behavior of $F(x, y)$ on the sides of the square. We note that the equation holds:

$$F(x, y)=\int_0^1 F(x, r) F(r, y)\, dr. \tag{10}$$

If $F(x, y)$ vanishes at a boundary point, say $(0, b)$, we have

$$0=\int_0^1 F(0, r) F(r, b)\, dr,$$

and as the integrand is never negative, it must vanish for all r. If $0<b<1$, $F(r, b)>0$ for all interior r, and $F(0, r)=0$. If $b=0$, the integrand is a square, and the same conclusion follows. If $b=1$, either one of the factors must vanish on the whole open interval, or one factor must vanish at some points, and the other at all the rest at least. But the last alternative is impossible, since (10) will show that if $F(x, y)$ is positive at one boundary point, it will be positive on the whole open side. We conclude that if $F(x, y)$ vanishes at any boundary point between corners, it vanishes on the whole side corresponding, and by symmetry, on a second side. The cases, then, are: I, $F(x, y)>0$ on the closed square; II, $F(x, y)>0$ on the open square, but vanishing on the whole boundary; and III, $F(x, y)>0$ on the open square and on two symmetric sides, but vanishing on the other two.

If b is any interior point, $F(x, b)$ is a harmonic, since (9) shows it to be a linear homogeneous combination of harmonics with constant coefficients. It may then be normalized and considered one of the harmonics $\phi_i(x)$, since if the functions of (9) be subjected to an orthogonal transformation, its form remains unaltered, the harmonics remaining normed and orthogonal. We may therefore identify $\phi_0(x)$ with $F(x, b)/\sqrt{\int_0^1 F^2(x, b)\, dx}$.

CASE I. Here $\phi_0(x)>0$ on the closed interval $(0, 1)$. If there is a second function $\phi_1(x)$, its ratio to $\phi_0(x)$ is continuous on the closed interval, and hence attains its maximum, M, say for $x=a$. Then $M\phi_0(x)-\phi_1(x)\geqq 0$ on $(0, 1)$. But as

$$M\phi_0(x)-\phi_1(x)=\int_0^1 F(x, y)\,[M\phi_0(y)-\phi_1(y)]\, dy, \tag{11}$$

we have, for $x=a$, a contradiction, since the integrand is continuous, has positive elements, and is never negative, while the left-hand side vanishes.

CASE II. Here, $F(x, y)$ being uniformly continuous, and vanishing on the whole boundary, it is possible, given $\varepsilon>0$, to find $\delta>0$, such that $0\leqq F(x, y)<\delta$ for $0\leqq x\leqq\delta$, and all y, and for $1-\delta\leqq x\leqq 1$ and all y, and for the corresponding regions formed by interchanging x and y. A similar condition holds for all harmonics as a consequence. For, for $0\leqq x\leqq\delta$, or $1-\delta\leqq x\leqq 1$, $\phi_i(x) =\int_0^1 F(x, y)\phi_i(y)\,dy$ is less in absolute value than

$$\varepsilon\int_0^1|\phi_i(y)|\,dy\leqq\varepsilon\sqrt{\int_0^1\phi_i^2(y)\,dy}=\varepsilon.$$

Now, if there be a second harmonic, the ratio $\phi_1(x)/\phi_0(x)$ is continuous on the closed interval $(\delta, 1-\delta)$, and attains its maximum M, say for $x=a$ within this interval. Then $M\phi_0(x)-\phi_1(x)\geqq 0$ on $(\delta, 1-\delta)$, and on the rest of $(0, 1)$, since $\phi_0(x)\geqq 0$, $M\phi_0(x)-\phi_1(x)>-\varepsilon$. But $M\phi_0(x)-\phi_1(x) =\int_0^1 F(x, y)\,[M\phi_0(y)-\phi_1(y)]\,dy=\int_\delta^{1-\delta}+\int_0^\delta+\int_{1-\delta}^1$, and as the first integral is never negative, we conclude that for $0\leqq x\leqq\delta$, or $1-\delta\leqq x\leqq 1$, $M\phi_0(x) -\phi_1(x)>-2\delta\varepsilon^2$. If the process is repeated, we see that this function is greater than $-(2\delta\varepsilon)^2\varepsilon$, and so on, so that the lower limit of $M\phi_0(x)-\phi_1(x)$ can only be 0 on the whole interval $(0, 1)$. We may then complete the reasoning as in Case I.

CASE III. A combination of the methods employed in the first two cases leads to the same result, that there can not be a second harmonic.

Thus $\phi_0(x)$, except for a multiplicative constant, is the only harmonic of $K_1(x, y)$ of frequency 1, and therefore the only harmonic of $K(x, y)$ of frequency $+1$ or -1. We have seen that it does not vanish within $(0, 1)$. Since it satisfies the integral equation (1) its frequency must be $+1$, because the integral and the left member of the equation have the same signs. This completes the proof of the lemma.

It is important for what follows to notice that though the lemma is proven for a kernel in two variables, the proof can be easily adapted to a real, continuous kernel symmetric in two sets of variables.

The lemma has established the property (D) for the harmonics $\phi_i(x)$ for $n=0$. To extend the argument to all values of n, we proceed to derive integral equations which the determinants $D_1, D_2, \ldots$ satisfy, and apply the lemma to their kernels.

An abbreviation will be helpful. Let ξ stand for the aggregate of variables $x_0, x_1, \ldots, x_n$; η for $y_0, y_1, \ldots, y_n$; and let us write $\varkappa(\xi, \eta)$ for $K\begin{pmatrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_n \end{pmatrix}$; and $\Delta(\xi)$ for $D_n(x_0, x_1, \ldots, x_n)$. Let S stand for the region $0 \leqq x_i \leqq 1$, $i=0, 1, \ldots, n$, and by $\int_S f(\xi)\, d\xi$ let us understand

$$\int_0^1 \int_0^1 \cdots \int_0^1 f(x_0, x_1, \ldots, x_n)\, dx_0, dx_1, \ldots, dx_n.$$

4. *The Integral Equation for* $\Delta(\xi)$.

We apply to $\Delta(\eta)\varkappa(\xi, \eta)$ Lagrange's product formula:

$$\Delta(\xi)\varkappa(\xi,\eta) = \begin{vmatrix} \sum_0^n \phi_0(y_i)K(x_0, y_i), & \sum_0^n \phi_0(y_i)K(x_1, y_i), & \ldots, & \sum_0^n \phi_0(y_i)K(x_n, y_i) \\ \sum_0^n \phi_1(y_i)K(x_0, y_i), & \sum_0^n \phi_1(y_i)K(x_1, y_i), & \ldots, & \sum_0^n \phi_1(y_i)K(x_n, y_i) \\ \cdots & \cdots & \cdots & \cdots \\ \sum_0^n \phi_n(y_i)K(x_0, y_i), & \sum_0^n \phi_n(y_i)K(x_1, y_i), & \ldots, & \sum_0^n \phi_n(y_i)K(x_n, y_i) \end{vmatrix}$$

$$= \Sigma \begin{vmatrix} \phi_0(y_{i_0})K(x_0, y_{i_0}), & \phi_0(y_{i_1})K(x_1, y_{i_1}), & \ldots, & \phi_0(y_{i_n})K(x_n, y_{i_n}) \\ \phi_1(y_{i_0})K(x_0, y_{i_0}), & \phi_1(y_{i_1})K(x_1, y_{i_1}), & \ldots, & \phi_1(y_{i_n})K(x_n, y_{i_n}) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_n(y_{i_0})K(x_0, y_{i_0}), & \phi_n(y_{i_1})K(x_1, y_{i_1}), & \ldots, & \phi_n(y_{i_n})K(x_n, y_{i_n}) \end{vmatrix},$$

where in the summation $i_0, i_1, \ldots, i_n$ take on all values from 0 to n. However, if any two of the indices are equal, the corresponding determinant is 0, two columns becoming proportional, so that the indices may be restricted to the $(n+1)!$ permutations of the numbers $0, 1, \ldots, n$. Thus we have $(n+1)!$ terms, which, after integration over the field S, all become equal, namely to

$$\begin{vmatrix} \phi_0(x_0)/\lambda_0, & \phi_0(x_1)/\lambda_0, & \ldots, & \phi_0(x_n)/\lambda_0, \\ \phi_1(x_0)/\lambda_1, & \phi_1(x_1)/\lambda_1, & \ldots, & \phi_1(x_n)/\lambda_1, \\ \cdots & \cdots & \cdots & \cdots, \\ \phi_n(x_0)/\lambda_n, & \phi_n(x_1)/\lambda_n, & \ldots, & \phi_n(x_n)/\lambda_n. \end{vmatrix}$$

Hence the required integral equation for $\Delta(\xi)$ is

$$\chi(\xi) = \mu \int_0^1 \chi(\eta)\varkappa(\xi, \eta)\, d\eta, \tag{12}$$

the solution $\Delta(\xi)$ corresponding to $\mu = (\lambda_0\lambda_1 \ldots \lambda_n)/(n+1)!$.

Various other solutions are obtained by forming determinants of sets of $n+1$ of the functions $\phi_i(x)$, other than the first $n+1$, the frequencies being the products of the corresponding frequencies of the $\phi_i(x)$ divided by $(n+1)!$

In order to apply the lemma of the last paragraph, it is necessary to know that the frequency of $\Delta(\xi)$ is the one of smallest absolute value of the kernel $\varkappa(\xi, \eta)$. To do this, we shall show that $\varkappa(\xi, \eta)$ has no other frequencies than those of the form $= \pm (\lambda_{i_0}\lambda_{i_1} \ldots \lambda_{i_n})/(n+1)!$. This is the subject of the next paragraph.

5. *The Frequencies of* $\varkappa(\xi, \eta)$.

We shall here make use of the following fact: $\lambda_0, \lambda_1, \ldots,$ *being a set of real quantities, they are determined by the power sums* $s_{-1} = \Sigma 1/\lambda_i$, $s_{-2} = \Sigma 1/\lambda_i^2, \ldots,$ *in case these sums are absolutely convergent.* The truth of this fact follows from our knowledge that the power sums determine a transcendental integral function whose roots the λ_i are.* Let us now assume, the justification to follow later, that $K(x, y)$ is the first iterated kernel of a real continuous symmetric kernel. It will follow from (4) that the power sums all converge absolutely, and will be given by

$$s_{-j} = \int_0^1 K_{j-1}(r, r)\, dr = \sum_i 1/\lambda_i^j. \tag{13}$$

Under the assumption just made, $\varkappa(\xi, \eta)$ will also be the first iterated kernel of a real, symmetric, continuous kernel and the power sums of its frequencies, also absolutely convergent, will be given by

$$s'_{-j} = \int_s \varkappa_{j-1}(\rho, \rho)\, d\rho = \sum_i 1/\mu_i^j. \tag{14}$$

We are concerned with establishing the equations:

$$s'_{-j} = \sum_{i_0, i_1, \ldots i_n} \left[\frac{(n+1)!}{\lambda_{i_0}\lambda_{i_1} \ldots \lambda_{i_n}}\right]^j, \quad j = 1, 2, \ldots\ldots \tag{15}$$

In order to prove this formula for $j=1$, we introduce the notation p_n, defining it by the equation:

$$(n+1)!\, p_{n+1} = \int_S K\begin{pmatrix} r_0, r_1, \ldots, r_n \\ r_0, r_1, \ldots, r_n \end{pmatrix} dr_0 dr_1 \ldots dr_n$$

$$= \int_0^1 \int_0^1 \ldots \int_0^1 \begin{vmatrix} K(r_0, r_0), K(r_0, r_1), \ldots, K(r_0, r_n) \\ K(r_1, r_0), K(r_1, r_1), \ldots, K(r_1, r_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ K(r_n, r_0), K(r_n, r_1), \ldots, K(r_n, r_n) \end{vmatrix} dr_0 dr_1 \ldots dr_n, \tag{16}$$

and shall prove that p_n is the sum of the products of the reciprocals of the λ_i n at a time. It will be noticed that the terms of the expanded determinant

* Log $P = \log (1 - z/\lambda_0)(1 - z/\lambda_1)(1 - z/\lambda_2) \ldots$ has these sums, apart from numerical factors, as coefficients, when expanded in a power series.

contain only products of the form $K(r_{i_1}, r_{i_2})K(r_{i_2}, r_{i_3}) \ldots K(r_{i_{k-1}}, r_{i_k})K(r_{i_k}, r_{i_1})$, which, upon integration becomes $\int_0^1 K_{k-1}(r, r)\,dr, = s_{-k}$. Fixing our attention upon those products containing a factor from the first row and one from the first column, we easily verify the following recursion formula for p_{n+1}:

$$(n+1)p_{n+1} = s_{-1}p_n - s_{-2}p_{n-1} + s_{-3}p_{n-2} - \ldots + (-1)^n s_{-1}.$$

But this is the recursion formula for the product sums in terms of the power sums given in the theory of algebraic equations, and applicable here because of the convergence of the power sums. And as (16) also gives $p_1 = s_{-1}$, it is clear that p_n is the product sum stated.*

On the other hand, $(n+1)!\,p_{n+1}$, by (16) is $\int_S \varkappa(\rho, \rho)\,d\rho = s'_{-1}$, so that (15) is proved for $j=1$.

To extend the proof is not difficult. We compute the value of the integral $\varkappa_1(\xi, \eta) = \int_S \varkappa(\xi, \rho)\varkappa(\rho, \eta)\,d\eta$, just as we did $\int_S \Delta(\eta)\varkappa(\xi, \eta)\,d\eta$ in the last paragraph, obtaining

$$\varkappa_1(\xi, \eta) = (n+1)!\begin{vmatrix} K_1(x_0, y_0), & K_1(x_0, y_1), & \ldots, & K_1(x_0, y_n) \\ K_1(x_1, y_0), & K_1(x_1, y_1), & \ldots, & K_1(x_1, y_n) \\ \cdots & \cdots & \cdots & \cdots \\ K_1(x_n, y_0), & K_1(x_n, y_1), & \ldots, & K_1(x_n, y_n) \end{vmatrix}, \qquad (17)$$

This shows that the power sum s'_{-2} may be computed just as was s'_{-1}, only replacing $K(x, y)$ by $K_1(x, y)$ and multiplying by $(n+1)!$. But the effect of replacing $K(x, y)$ by $K_1(x, y)$ is to replace the numbers λ_i by their squares. Thus the formula (15) is demonstrated for $j=2$. It is clear that the method is general, and we may regard (15) as true for all positive integers j.

Reverting to our assumption that $K(x, y)$ was the first iterated kernel of a continuous symmetric kernel, we note that should there be doubt of this, we may, in the above reasoning, begin with $K_1(x, y)$ instead of $K(x, y)$. Equation (17) then shows that $\varkappa(\xi, \eta)$ would be replaced by $\varkappa_1(\xi, \eta)/(n+1)!$, and instead of the frequencies of $\varkappa(\xi, \eta)$ being identified with the products $(\lambda_{i_0}\lambda_{i_1} \ldots \lambda_{i_n})/(n+1)!$, we should only have their squares identified with the squares of these products. But this is sufficient for the purposes in hand, and gives the result announced at the close of the last paragraph.

* The result has an interesting bearing on the Fredholm series for symmetric kernels, for p_n is the coefficient of $(-\lambda)^n$ in the series $d(\lambda) = 1 - \frac{\lambda}{1!}\int_0^1 K(r, r)\,dr + \frac{\lambda^2}{2!}\int_0^1\int_0^1 K\begin{pmatrix} r_0, & r_1 \\ r_0, & r_1 \end{pmatrix} dr_0\,dr_1 - \ldots$, whose roots are the frequencies of $K(x, y)$. It shows that if the power sums are absolutely convergent, as they will be, for example, if $K(x, y)$ is the first iterated kernel of a real, symmetric, continuous kernel, the transcendental integral function $d(\lambda)$ *contains as factor no transcendental integral function without zeros.*

6. *The Property* (D).

Evidently the sign of $\Delta(\xi)$ is not constant on S, as an interchange of two arguments changes the sign of $\Delta(\xi)$. But we are concerned only with showing it different from 0 on R. We therefore substitute in the equation (12), the field R for the field S. In doing so, we note that $\Delta(\eta)$ and $\varkappa(\xi,\eta)$ are alternating functions on S for η, and that therefore their product is symmetric. Hence the field R is one of $(n+1)!$ symmetric sub-fields for the integral, corresponding to one of the $(n+1)!$ orders of the arguments y. Hence $\int_S \Delta(\eta)\varkappa(\xi,\eta)\,d\eta=(n+1)!\int_R \Delta(\eta)\varkappa(\xi,\eta)\,d\eta$, so that $\Delta(\xi)$ now satisfies the equation

$$\chi(\xi)=\nu\int_R \chi(\eta)\varkappa(\xi,\eta)\,d\eta \tag{18}$$

for $\nu=\lambda_0\lambda_1\ldots\lambda_n$. This equation will have the same harmonics as (12), and its frequencies will evidently be those of (12) multiplied by $(n+1)!$.

The hypothesis (K) on $K(x,y)$ includes the hypothesis (K) for $n=0$ on $\varkappa(\xi,\eta)$, and the lemma of paragraph 3 applies. As the frequency of least absolute value of $\varkappa(\xi,\eta)$ is $\pm\lambda_0\lambda_1\ldots\lambda_n$, since this is the smallest (or one of the smallest, in case several are equal) product of $n+1$ of the λ_i in absolute value, the single harmonic belonging to this frequency must be $\Delta(\xi)$. And this one harmonic does not vanish on R. Hence, by proper choice of the sign of $\phi_n(x)$, we may conclude $\Delta(\xi)=D_n(x_0,x_1,\ldots,x_n)>0$ on R, and the property (D) is thus generally established for the harmonics of $K(x,y)$.

7. *A Generalization.*

As, in the foregoing, we have used the hypothesis (K) only for one value of n, the following theorem follows:

If the real, continuous, symmetric kernel $K(x,y)$ *satisfies the hypothesis* (K) *for* $n=n_1, n_2, \ldots,$ *then the harmonics of* $K(x,y)$ *satisfy the condition* (D) *for* $n=n_1, n_2, \ldots$.

In the next number of this Journal I shall establish the property (D) for the orthogonal function sets arising from ordinary linear homogeneous differential equations of the second order.

COLUMBIA, MO., *October* 20, 1917.

Complete Systems of Concomitants of the Three-Point and the Four-Point in Elementary Geometry.

BY CHARLES HENRY RAWLINS, JR.

INTRODUCTION.

In this discussion, two point-sets, containing three and four points respectively, are subjected to three transformations of elementary geometry; and complete systems of invariants and covariants, corresponding to the respective transformations, are derived. In the process we obtain some interesting geometric applications of the theory of binary forms and of symmetric functions.

PART I.

INVARIANTS UNDER TRANSLATION.

Section (a): *The Three-Point.*

Consider first the three-point under translation. Let the points have complex coordinates α, β, γ, respectively, the roots of a cubic

$$a_0x^3+3a_1x^2+3a_2x+a_3=0.$$

Since translation is effected by adding to each root the same vector p, the invariants of the three-point under translation are the geometric equivalents of the invariants of the cubic under the transformation

$$x'=x+p.$$

Invariants under such a transformation are known, in the theory of binary forms, as *seminvariants.* Their complete system for the cubic is known* to consist of the coefficients of the cubic when so transformed that its second term vanishes, and the discriminant

$$\Delta\equiv(a_0a_3-a_1a_2)^2-4(a_0a_2-a_1^2)(a_1a_3-a_2^2).$$

Substituting $x-(a_1/a_0)$ for x in the cubic, we obtain

$$A_0x^3+3A_2x+A_3=0,$$

* Elliott, "Algebra of Quantics," pp. 140, 162.

the form desired, where, except for factor a_0,

$$A_0=a_0,\quad A_2=a_0a_2-a_1^2,\quad A_3=a_0^2a_3-3a_0a_1a_2+2a_1^3.$$

Connecting these is the syzygy

$$A_0^2\Delta=4A_2^3+A_3^2$$

However, without loss of generality, we can consider A_0 (or a_0) as having the fixed value unity, in which case Δ is expressed *integrally* in terms of A_2 and A_3. Therefore

The complete system of invariants of the three-point under translation consists of A_2 and A_3.

Since the *shape* of the three-point is an invariant property under translation, the vanishing of A_2 or A_3 indicates some condition on the shape.

Vanishing of A_2:—If $A_2=0$, the cubic is

$$x^3+A_3=0,$$

with roots in ratio $1:\omega:\omega^2$, where ω is one of the complex cube roots of unity. The points form, therefore, the vertices of an equilateral triangle.

Vanishing of A_3:—If $A_3=0$, the cubic is

$$x^3+3A_2x=0$$

with roots in ratio $0:1:-1$. Hence the points are collinear, with one midway between the others.

A_1 being absent, the sum of the roots is zero and therefore the origin is at the centroid.

Section (b): The Four-Point.

Similarly, *the complete system of seminvariants of the four-point*

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,$$

consists of *

$$A_0=a_0(=1),\quad A_2=a_0a_2-a_1^2,\quad A_3=a_0^2a_3-3a_0a_1a_2+2a_1^3,$$
$$I\ =a_0a_4-4a_1a_3+3a_2^2,\quad J=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$$

However, since

$$A_4=a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4=a_0^2I-3A_2^2,$$

we will use it instead of I.

Again the origin is at the centroid, and the vanishing of an invariant is a condition on the *shape* of the configuration.

Vanishing of A_2:—Let $\alpha,\beta,\gamma,\delta$ be the roots of

$$A_0x^4+6A_2x^2+4A_3x+A_4=0.$$

---

* Elliott, *supra*, p. 170.

Then, if $A_2=0$, $\Sigma\alpha\beta=0$; so that, since $\overset{4}{\Sigma}\alpha=0$, then $\overset{4}{\Sigma}\alpha^2=0$. Refer the system to rectangular axes through the origin, so that $\alpha=X_1+iY_1$, $\beta=X_2+iY_2$, etc. Then

$$\sum_{j=1}^{4}(X_j+iY_j)^2=0.$$

Expanding and equating real parts,

$$\Sigma X^2-\Sigma Y^2=0.$$

But

$$\Sigma X^2+\Sigma Y^2=\Sigma R^2,$$

where the R's are distances from the origin to the respective points, constants in the present discussion. Therefore

$$\Sigma X^2=1/2\Sigma R^2=\text{constant}. \tag{1}$$

If the points are of unit mass, ΣX is the moment of inertia of the system about the Y-axis. But, the direction of this axis having been arbitrarily chosen, (1) tells us that the moment of inertia is the same for all axes through the centroid, thus making the "ellipse of inertia"* a circle.

Vanishing of A_3:—If $A_3=0$, the quartic is

$$x^4+6A_2x^2+A_4=0,$$

with factors x^2-m_1, x^2-m_2, and roots $\pm\sqrt{m_1}$, $\pm\sqrt{m_2}$. Hence the points form the vertices of a parallelogram.

Vanishing of A_4:—If $A_4=0$, one root of the quartic is zero. Hence one point lies at the centroid of the other three.

Vanishing of J:—The vanishing of J is known to be the condition that the four roots form harmonic pairs; or, geometrically, that the four points lie on one of a set of three mutually orthogonal circles, at its intersections with the other two.

Since *infinity* is unaltered by a finite translation, it has not been necessary, in the foregoing, to consider it as part of the apparatus employed.

PART II.

Monogenic Concomitants in the Complex Plane.

Section (a): General Theory.

We will consider next the monogenic concomitants of the point-sets in the complex plane, that is, concomitants not involving the conjugates of any of

---

* Routh, "Rigid Dynamics," Chap. I.

the complex quantities used. Then the most general (linear) transformation possible is

$$x' = (ax+b)/(cx+d)$$

the product of an even number of inversions. Since infinity is not, in general, a fixed point of this transformation, it must be considered explicitly, that is:

We must discuss not merely the concomitants of a point-set, *but the concomitants of the point at infinity and the point-set.* Hence we use the theory of a binary cubic (quartic) and a linear form. From this theory we learn * that *the complete system of the cubic (quartic) and the linear form consists of the linear form, the complete system of the cubic (quartic) and the polars of this system with respect to the linear form.*

The linear form being the equation of infinity, its polar operator, under the present notation, is simply d/dx.

Section (b): The Three-Point.

The complete system of the cubic consists of †

the cubic itself: $C = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$,

the Jacobian: $G = A_3x^3 + \dots,$

the Hessian: $H = A_2x^2 + \dots,$

the discriminant: $\Delta = (a_0a_3 - a_1a_2)^2 - 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2)$.

Successive derivatives of these are (neglecting constant factors):

$$\begin{array}{lll} C' = a_0x^2 + 2a_1x + a_2, & C'' = a_0x + a_1, & C''' = a_0 \ \text{(negligible)}, \\ G' = A_3x^2 + \dots, & G'' = A_3x + \dots, & G''' = A_3, \\ H' = A_2x + \dots, & H'' = A_2. & \end{array}$$

As in Part I, Section (a), Δ is expressible integrally in terms of A_2 and A_3.

Complete System.—Thus the complete system contains

two cubics: $C,\ G,$

three quadratics: $C',\ G',\ H,$

three linear forms: $C'',\ G'',\ H';$

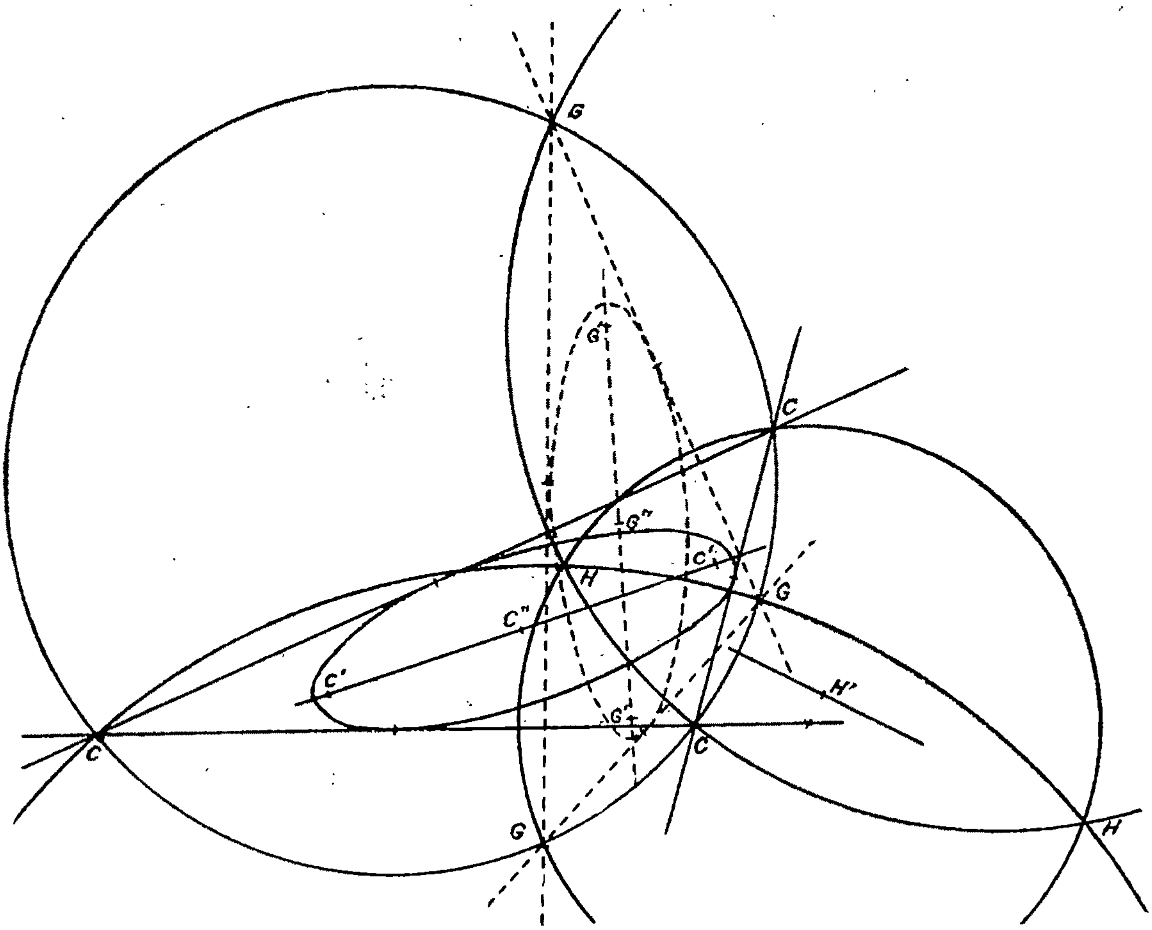
two invariants: $G''' = A_3,\ H'' = A_2.$

Geometric Equivalents.—We will state, without demonstration, the geometric equivalents (see figure).

* Grace & Young, "Algebra of Invariants," 138 A.

† Salmon, "Higher Algebra"; Elliott and Grace & Young, *supra.*

The G-points are cut out of the circle on the C-points by means of the three Apollonian circles, that is, circles each on one C-point and *drawn about* the other two. The Apollonian circles are members of a pencil whose fixed points are the H-points. C' are the foci of the ellipse inscribed in the C-triangle with its center at the centroid, and C'' is the centroid. G' and G'' are similarly related to the G-triangle. H' is midway between the H-points. If $A_2=0$, an H-point is at infinity. If $A_3=0$, a G-point is at infinity.



Section (c): The Four-Point.

The complete system of the quartic consists of

the quartic itself: $Q=a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4$,

the Jacobian: $G=A_3x^6+\ldots.$,

the Hessian: $H=A_2x^4+\ldots.$,

two invariants:

$$I=a_0a_4-4a_1a_3+3a_2^2,\quad J=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$$

Complete System.—Taking derivatives, the complete system of the four-point is

one sextic:	G,
one quintic:	G',
three quartics:	Q, G'', H,
three cubics:	Q', G''', H',
three quadratics:	Q'', $G^{(4)}$, H'',
three linear forms:	Q''', $G^{(5)}$, H''',
four invariants:	$G^{(6)}=A_3$; $H^{(4)}=A_2$, I, J.

Geometric Equivalents.—We will define a few of the geometric equivalents of these forms. G is the double points of the involutions formed by taking the Q-points in pairs. The derived forms can be interpreted by the following theorem:*

Given $\phi(p_1, p_2, \ldots, p_r)=0$, homogeneous in the p's, as the equation of a curve, where the p's are the distances from given points $a_1, a_2, \ldots, a_r$, respectively, to a line of the curve, the foci of the curve are the roots of $\phi(x-a_1, x-a_2, \ldots, x-a_r)=0$.

Suppose now n points, $a_1, a_2, \ldots, a_n$, given by

$$f=(x-a_1)(x-a_2)\ldots(x-a_n)=0.$$

The derived equation can be put in the form

$$f'=\frac{1}{x-a_1}+\frac{1}{x-a_2}+\ldots+\frac{1}{x-a_n}=0.$$

By the theorem, f' gives the foci of the curve

$$\sum_{i=1}^{n} 1/p_i=0,$$

which is of class $n-1$, is on the $n(n-1)/2$ joins of the n points and touches each join at its mid-point.

Hence G' is the foci of such a curve on the joins of the G-points, and similarly for the other derived forms.

If $A_2=0$, an H-point is at infinity. If $A_3=0$, a G-point is at infinity.

---

* F. Morley, Lectures 1915–16, Johns Hopkins University.

where the A's are expressions in λ, a, and x, consistent with the three-term form in which used.

Eliminating the A's, we obtain

$$\begin{vmatrix} (Ax^n) & x_0^n & x_1^n & x_2^n \\ (Ax^{n-1}) & x_0^{n-1} & x_1^{n-1} & x_2^{n-1} \\ (Ax^{n-2}) & x_0^{n-2} & x_1^{n-2} & x_2^{n-2} \\ (Ax^{n-3}) & x_0^{n-3} & x_1^{n-3} & x_2^{n-3} \end{vmatrix} = 0,$$

which, after the removal of certain factors, gives

$$(Ax^n) = (x)(Ax^{n-1}) - (x_1x_2)(Ax^{n-2}) + x_0x_1x_2(Ax^{n-3}).$$

Repeating the process on (Ax^{n-1}), etc., we obtain finally (Ax^n) in terms of (Ax^2), (Ax), (A), (x), (x_1x_2) and $x_0x_1x_2$. Since the process applies equally well to the λ's and the a's.

(C) *The complete system will contain no symmetric functions in which an element occurs to a higher degree than the second.*

We state for reference the following special case of the above formula:

$$(Ax^3) = (x)(Ax^2) - (x_1x_2)(Ax) + x_0x_1x_2(A). \tag{1}$$

Since by (B) no term is to contain more than two elements of a kind, the total degree in x (λ or a) must not exceed four. But, if the function be $(A_0x_1^2x_2^2)$, of total degree 4 in x, we can use the important identities

$$x_ix_j = (x_1x_2) - (x)x_k + x_k^2 \qquad (i, j, k = 0, 1, 2;\ i \neq j \neq k), \tag{2}$$

and obtain

$$(A_0x_1^2x_2^2) = (A)(x_1x_2)^2 + (x)^2(Ax^2) + (Ax^4) - 2(x)(x_1x_2)(Ax)$$
$$-2(x_1x_2)(Ax^2) - 2(x)(Ax^3),$$

in which (Ax^4) and (Ax^3) are reducible by (C).

A function of total degree 3 in x is of the type $\Sigma A_0x_1^2x_2$. This necessarily contains six terms and is therefore excluded by (A). Hence:

(D) *The complete system will contain no symmetric functions in which the total degree in λ, a, or x is greater than* 2, *except the special functions* $\lambda_0\lambda_1\lambda_2$, $a_0a_1a_2$, $x_0x_1x_2$.

For reference, we restate (2) in different form:

$$x_i^2 = x_jx_k + (x)x_i - (x_1x_2) \qquad (i, j, k = 0, 1, 2;\ i \neq j \neq k). \tag{3}$$

A function can contain x to total degree 2 in two ways:

$$(Ax^2) \quad \text{or} \quad (A_0x_1x_2).$$

But, by (2),

$$(A_0x_1x_2) = (A)(x_1x_2) - (x)(Ax) + (Ax^2).$$

Hence:

(E) *As members of the complete system of symmetric functions, the forms* $(A_0x_1x_2)$ *and* (Ax^2) *(similarly for the λ's and a's) are mutually exclusive.*

We will adopt the form $(A_0x_1x_2)$.

Denoting by s_{pqr} a three-term symmetric function of total degree p in λ, q in a, and r in x; principles (A) to (E), inclusive, leave for individual consideration

s_{001}	s_{010}	s_{100}
s_{002}	s_{020}	s_{200}
s_{003}	s_{030}	s_{300}
s_{011}	s_{101}	s_{110}
s_{012}	s_{102}	s_{120}
s_{021}	s_{201}	s_{210}
s_{022}	s_{202}	s_{220}
s_{111}		
s_{112}	s_{121}	s_{211}
s_{122}	s_{212}	s_{221}
s_{222}		

We need consider only the first column in detail, because the others are obtained from it by interchange of letters.

$s_{001}=(x)$ and $s_{002}=(x_1x_2)$ are obviously irreducible.

$s_{003}=x_0x_1x_2$, $s_{011}=(ax)$, $s_{012}=(a_0x_1x_2)$, and $s_{021}=(a_1a_2x_0)$ are found to be irreducible by a test which will be illustrated by use on $s_{022}=(a_1a_2x_1x_2)$.

This, if reducible, will be a sum of products of irreducible functions, each product of degree 022. Assume then

$$(a_1a_2x_1x_2)=k(a)(a_0x_1x_2)+l(x)(a_1a_2x_0)+m(a_1a_2)(x_1x_2)+n(ax)^2 \\ +p(a)^2(x_1x_2)+q(x)^2(a_1a_2)+r(a)(x)(ax)+s(a)^2(x)^2.$$

Then, by substituting sets of numerical values for the a's, and x's we obtain a sufficient number of equations, simultaneous in k, l, m, etc., to solve for their values. The operation is greatly simplified by a selection of numbers which causes several terms of the assumed identity to vanish.

For instance, substituting

$$1, -1, 0;\ 1, 1, -2 \text{ for } a_0, a_1, a_2;\ x_0, x_1, x_2,$$

respectively; (a), (ax), and (x) vanish, and we obtain

$$-1=3m, \text{ whence } m=-1/3.$$

Similarly, from

$$1, -1, 0;\ 1, -1, 0; \quad n=1/3,$$
$$1, \ \ 0, 0;\ 1, -1, 0; \quad p=1/3, \text{ etc.}$$

As a result

$$\text{(F)} \quad s_{022}=(a_1a_2x_1x_2)=-1/3\,(a)\,(a_0x_1x_2)-1/3\,(x)\,(a_1a_2x_0) -1/3\,(a_1a_2)(x_1x_2)+1/3\,(ax)^2 +1/3\,(a)^2(x_1x_2)+1/3\,(x)^2(a_1a_2)-1/3\,(a)(x)(ax),$$

the correctness of which has been tested by various numerical substitutions.

If such substitutions fail to satisfy the expression, or if contradictions arise in solving for the coefficients, the assumption of an identity is false and the function is irreducible. As an illustration, assume

$$s_{012}=(a_0x_1x_2)=k\,(x)\,(ax)+l\,(a)\,(x_1x_2)+m\,(a)\,(x)^2.$$

From $\quad 1, -1, 0;\ 1, \ 0, 0; \quad k=0,$

and from $\quad 1, \ 1, 0;\ 1, -1, 0; \quad l=0,$

whence $\quad (a_0x_1x_2)=m\,(a)\,(x)^2,$

which is obviously untrue, whatever the value of m.

In this way s_{003}, s_{011}, s_{021} and $s_{111}=(\lambda ax)$ are proved irreducible. Similar methods show that

$$\text{(G)} \quad s_{112}=(\lambda_0a_0x_1x_2)=1/3\,(\lambda)\,(a_0x_1x_2)+1/3\,(a)\,(\lambda_0x_1x_2) -1/3\,(x)\,(\lambda ax)+2/3\,(\lambda a)\,(x_1x_2) +1/3\,(\lambda x)(ax)-1/3\,(\lambda)\,(a)\,(x_1x_2),$$

which stands the test for an identity.

(H) $s_{122}=(\lambda_0a_1a_2x_1x_2)$ reduces by substituting a_1a_2 for a_0, a_2a_0 for a_1, a_0a_1 for a_2 in (G). Similarly,

(K) $s_{222}=(\lambda_1\lambda_2a_1a_2x_1x_2)$ reduces by substituting $\lambda_1\lambda_2$ for λ_0, etc., in (H).

Complete System.—We have, therefore, the following complete system of symmetric functions:

Of λ alone	s_{100}	s_{200}	s_{300}
Of a alone	s_{010}	s_{020}	s_{030}
Of x alone	s_{001}	s_{002}	s_{003}
Of a and x	s_{011}	s_{012}	s_{021}
Of λ and x	s_{101}	s_{102}	s_{201}
Of λ and a	s_{110}	s_{120}	s_{210}
Of λ, a and x	s_{111}		

Furthermore, the foregoing processes, (A) to (K) inclusive, enable us to express any given symmetric function in terms of the members of the complete system.

Section (d): Alternating Functions.

Using the same notation as in Section (*c*), the most general alternating function of three rows is

$$K=p_0q_1r_2+p_1q_2r_0+p_2q_0r_1-p_0q_2r_1-p_1q_0r_2-p_2q_1r_0.$$

This is the same as

$$|pqr|\equiv\begin{vmatrix} p_0 & q_0 & r_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}.$$

Taking K however in the form $\overset{3}{\Sigma}Ap$, where $A_0=q_1r_2-q_2r_1$, etc., it is easily proved that

(B′) *The complete system will contain no alternating functions with more than two λ's, two a's and two x's in a term.*

(C′) *The complete system will contain no alternating functions in which an element occurs to a degree higher than the second.*

Functions containing λ, a or x to total degree 4 can be reduced by (2), Section (*c*), to functions of total degree 2 in that quantity. Functions of total degree 3 in λ, a or x are not reducible by this means. Hence

(D″) *The complete system will contain no alternating functions in which the total degree in λ, a or x is greater than 3.*

(E) can be restated:

(E′) *As members of the complete system,* $\overset{3}{\Sigma}A_ix_1x_2$ *and* $\overset{3}{\Sigma}A_ix_0^2$ *(similarly for the λ's and a's) are mutually exclusive.*

A further important principle is obtained by considering the determinant

$$\begin{vmatrix} p_0 & q_0 & r_0 & 1/3 \\ p_1 & q_1 & r_1 & 1/3 \\ p_2 & q_2 & r_2 & 1/3 \\ (p) & (q) & (r) & 1 \end{vmatrix}.$$

This vanishes identically because the last row is the sum of the others. Hence, expanding:

$$|p\ q\ r|=1/3\,(p)\,|q\ r\ 1|-1/3\,(q)\,|p\ r\ 1|+1/3\,(r)\,|p\ q\ 1|.$$

This reduces any alternating function except those in which p, q or r equals unity. Therefore:

(L) *The complete system will contain no alternating functions except those whose determinants have a column (or row) of 1's.*

Such functions, of total degree 3 in x, are of the type

$$|ux^2\ vx\ 1|$$

where u and v are products of powers of λ and a. By (2), Section (c),

$$|ux^2\ vx\ 1| = (x_1x_2)\,|ux\ v\ 1| - (x)\,|ux\ v\ x| + |ux\ v\ x^2|.$$

Only the last function in this expansion contains x to total degree as high as 3. Applying (L) to this function,

$$|ux\ v\ x^2| = 1/3\,(ux)\,|v\ x^2\ 1| - 1/3\,(v)\,|ux\ x^2\ 1| + 1/3\,(x^2)\,|ux\ v\ 1|.$$

Likewise, in this expansion, only $|ux\ x^2\ 1|$ is of total degree 3 in x. Applying (2) to this,

$$|ux\ x^2\ 1| = (x_1x_2)\,|u\ x\ 1| + |u\ x\ x^2|.$$

Again, applying (L) to $|u\ x\ x^2|$,

$$|u\ x\ x^2| = (u)\,|1\ x\ x^2| - (x)\,|u\ x^2\ 1| + (x^2)\,|u\ x\ 1|.$$

Finally, $|1\ x\ x^2|$ is unaltered by (2), (3) or (L).

If u, v, or both equal unity, the process is merely shortened; the conclusion is the same, namely:

(D′) *The complete system will contain no alternating functions in which the total degree in λ, a or x is greater than 2, except possibly $|x^2\ x\ 1|$, $|\lambda^2\ \lambda\ 1|$ and $|a^2\ a\ 1|$.*

Consider the cases where at least two elements occur, each to total degree 2. If w and w' are powers of λ, we have the types

(a) $|wax\ \ w'ax\ \ 1|$, (b) $|wax^2\ \ w'a\ \ 1|$, (c) $|wa^2x^2\ \ w'\ \ 1|$,
(d) $|wa^2x\ \ w'x\ \ 1|$, (e) $|wa^2\ \ w'x^2\ \ 1|$.

By (2), (a) reduces to $|wa\ \ w'a\ \ x^2|$ and functions of lower degree.
By (3), (b) " " $|wa\ \ w'ax\ \ x|$ " " " " "
By (3), (c) " " $|wa^2\ \ w'x\ \ x|$ " " " " "
By (2), in terms of the element a,
(d) reduces to $|wx\ \ w'ax\ \ a|$ " " " " "
By (3), (e) " " $|wa^2x\ \ w'\ \ x|$ " " " " "

All of these results can be reduced by (L), except that of (e) when $w'=1$, thus giving $|wa^2x\ x\ 1|$. But, by (3) in terms of the element a, this reduces to $|wx\ ax\ a|$ (and functions of lower degree) and (L) reduces this form. Hence:

(M) *The complete system will contain no alternating functions in which more than one element occurs to total degree 2.*

We have remaining, for individual consideration, the types:

a_{001}	a_{010}	a_{100}
a_{002}	a_{020}	a_{200}
a_{003}	a_{030}	a_{300}
a_{011}	a_{101}	a_{110}
a_{012}	a_{102}	a_{120}
a_{021}	a_{201}	a_{210}
a_{111}		
a_{112}	a_{121}	a_{211}

a_{001} and a_{002} can be constructed only in forms which vanish identically. a_{003} has, by (D′), the one type, $|x^2\, x\, 1|$, which is obviously irreducible. a_{011} can appear only as $|x\, a\, 1|$ and is obviously irreducible. a_{012} appears either as $|x^2\, a\, 1|$ or $|x\, ax\, 1|$. The only possible assumption is

$$|x^2\, a\, 1| \;\text{ or }\; |x\, ax\, 1| = k(x)\,|x\, a\, 1|.$$

If $a_i = x_i$, we have

$$|x^2\, x\, 1| \;\text{ or }\; |x\, x^2\, 1| = k(x)\,|x\, x\, 1| = 0,$$

which is untrue. By (2),

$$|x\, ax\, 1| = |1\, a\, x^2| - (x)\,|1\, a\, x| - |x^2\, a\, 1| + (x)\,|x\, a\, 1|.$$

Hence, $|x^2\, a\, 1|$ can be taken as the irreducible type.

Interchanging a and x in this form, $a_{021} = |x\, a^2\, 1|$ is seen to be irreducible. a_{111} can appear as:

$$\text{(a)}\quad |x \quad \lambda a \quad 1| = \sum^{3} x_0(\lambda_1 a_1 - \lambda_2 a_2),$$

$$\text{(b)}\quad |\lambda x \quad a \quad 1| = \sum^{3} \lambda_0 x_0 (a_1 - a_2),$$

$$\text{(c)}\quad |ax \quad \lambda \quad 1| = \sum^{3} a_0 x_0 (\lambda_1 - \lambda_2).$$

The sum of (a) and (b) is

$$\begin{aligned}\sum^{3} x_0[a_1(\lambda_1+\lambda_0) - a_2(\lambda_2+\lambda_0)] &= \sum^{3} x_0[a_1((\lambda)-\lambda_2) - a_2((\lambda)-\lambda_1)] \\ &= (\lambda)\sum^{3} x_0(a_1 - a_2) - \sum^{3} x_0(\lambda_2 a_1 - \lambda_1 a_2) \\ &= (\lambda)\,|x\, a\, 1| - |x\, a\, \lambda|.\end{aligned}$$

Hence

$$|\lambda x\, a\, 1| = -|x\, \lambda a\, 1| + (\lambda)\,|x\, a\, 1| - |x\, a\, \lambda|,$$

the last term of which is reducible by (L).

Interchanging λ and a,

$$|ax\ \lambda\ 1| = -|x\ \lambda a\ 1| + (a)\,|x\ \lambda\ 1| - |x\ \lambda\ a|.$$

Thus, (b) and (c) are expressible in terms of (a) and functions of lower degree.

Assume (a):

$$|x\ \lambda a\ 1| = k(x)\,|\lambda\ a\ 1| + l(a)\,|x\ \lambda\ 1| + m(\lambda)\,|x\ a\ 1|.$$

Let $\lambda_i = x_i$, $a_i = x_i$, then

$$|x\ x^2\ 1| = k(x)\,|x\ x\ 1| + l(x)\,|x\ x\ 1| + m(x)\,|x\ x\ 1| = 0,$$

which is untrue. Hence

$a_{111} = |x\ \lambda a\ 1|$ is an irreducible function.

a_{112} can appear as

(a) $|x^2\ \ \lambda a\ 1|$, (b) $|\lambda ax\ \ x\ \ 1|$, (c) $|\lambda x^2\ a\ 1|$
(d) $|ax^2\ \lambda\ \ 1|$, (e) $|\lambda x\ \ ax\ 1|$.

By (2), (b) reduces to (a) and functions of lower degree.
By (3), (c) " " $|\lambda\ ax\ x|$ and functions of lower degree.
By (3), (d) " " $|a\ \lambda x\ x|$ " " " " "
By (2), (e) " " $|\lambda\ a\ \ x^2|$ " " " " "

Hence (a) is the one form requiring further examination. Adding this to (c), we obtain finally

(N) $$|x^2\ \lambda a\ 1| = -|\lambda x^2\ a\ 1| + (\lambda)\,|x^2\ a\ 1| - |x^2\ a\ \lambda|,$$

and the right-hand members are reducible.

Complete System.—We have, therefore, the following complete system of alternating functions:

Of λ alone	$a_{300} = \lvert\lambda^2\ \lambda\ \ 1\rvert$.		
Of a alone	$a_{030} = \lvert a^2\ a\ \ 1\rvert$.		
Of x alone	$a_{003} = \lvert x^2\ x\ \ 1\rvert$.		
Of a and x	$a_{011} = \lvert x\ \ a\ \ 1\rvert$,	$a_{012} = \lvert x^2\ a\ \ 1\rvert$,	$a_{021} = \lvert x\ \ a^2\ 1\rvert$.
Of λ and x	$a_{101} = \lvert x\ \ \lambda\ \ 1\rvert$,	$a_{102} = \lvert x^2\ \lambda\ \ 1\rvert$,	$a_{201} = \lvert x\ \ \lambda^2\ 1\rvert$.
Of λ and a	$a_{110} = \lvert\lambda\ \ a\ \ 1\rvert$,	$a_{120} = \lvert\lambda\ \ a^2\ 1\rvert$,	$a_{210} = \lvert\lambda^2\ a\ \ 1\rvert$.
Of λ, a and x	$a_{111} = \lvert x\ \ \lambda a\ 1\rvert$.		

Furthermore, the foregoing processes enable us to express any given alternating function in terms of the members of the complete systems of symmetric and of alternating functions.

Section (e): Functions of n Rows of Elements.

The operations explained in Sections (c) and (d) are not confined to functions of three rows only. It is easily seen that theorems (A) to (E) inclusive and (A′) to (M) inclusive can be re-stated, without material alteration, for a function of n rows of three quantities each.

Consider a symmetric function of four rows, β, λ, a, x. The complete system will contain the irreducible functions of λ, a, x, and similar functions of β, a, x, etc. In addition, we must examine

$$\begin{array}{lll} s_{1111} & & \\ s_{1112} & s_{1121}, \text{ etc.,} & \\ s_{1122} & s_{1212}, \text{ etc.,} & \\ s_{1222}, \text{ etc.,} & & \\ s_{2222} & & \end{array}$$

$s_{1112}=(\beta_0\lambda_0 a_0 x_1 x_2)$ reduces by substituting $\beta_i\lambda_i$ for λ_i in (G). Similar methods apply to s_{1122}, s_{1222}, s_{2222}. Assume

$$s_{1111}=(\beta\lambda ax)=k(\beta)(\lambda ax)+m(\lambda)(\beta ax)+\ldots\ldots$$

Setting $\beta_0=\beta_1=\beta_2=1$, and collecting terms, we have

$$(\lambda ax)=k'(\lambda ax)+m'(\lambda)(ax)+n'(a)(\lambda x)+p'(x)(\lambda a)+q'(\lambda)(a)(x).$$

If $k'\neq 1$, this gives an expansion for (λax), which is impossible. If $k'=1$, we have

$$m'(\lambda)(ax)+n'(a)(\lambda x)+p'(x)(\lambda a)+q'(\lambda)(a)(x)=0.$$

Let $\lambda_0=a_0=1$, $\lambda_1=a_1=-1$, $\lambda_2=a_2=0$, so that $(\lambda)=(a)=0$. Then

$$2p'(x)=0, \quad \text{whence} \quad p'=0.$$

Similarly, $m'=n'=q'=0$, so the identity is not true for finite coefficients. Therefore:

Of symmetric functions of finite degree in all four rows, s_{1111} alone is irreducible.

Likewise

$a_{1111}=|\beta\lambda a\ x, 1|$ *is the only irreducible alternating function of finite degree in each of four rows.*

Similar statements are readily seen to hold true for 5, 6, ..., n rows.

Section (f): Geometric Classification.

The Three-Point.—The complete system for the three-point consists of those forms, deduced in Sections (c) and (d), which do not contain a. The subscripts denoting degrees in λ and x, respectively, we have

	Symmetric	Alternating	Total
Cubics	s_{03}	a_{03}	2
Conics	s_{02} s_{12}	a_{12}	3
Lines	s_{01} s_{11} s_{21}	a_{11} a_{21}	5
Invariants	s_{10} s_{20} s_{30}	a_{30}	4
Total	9	5	14

The Four-Point.—The complete system for the four-point consists of the entire list obtained in Sections (c) and (d). Rearranged according to degree in x, they are:

	Symmetric	Alternating	Total
Cubics	s_{003}	a_{003}	2
Conics	s_{002} s_{012} s_{102}	a_{012} a_{102}	5
Lines	s_{001} s_{011} s_{021} s_{201} s_{101} s_{111}	a_{011} a_{101} a_{111} a_{021} a_{201}	11
Invariants	s_{100} s_{010} s_{110} s_{210} s_{120} s_{020} s_{200} s_{030} s_{300}	a_{300} a_{030} a_{120} a_{210} a_{110}	14
Total	19	13	32

The members of the two systems are the concomitants of the three-point and the four-point under linear transformations sending the line at infinity into itself, either leaving the two points of the absolute fixed, or interchanging them.

Section (g): Syzygies.

It is not our purpose to derive a system of syzygies. However, it appears that the following method is useful in doing so. Take the determinant

$$\begin{vmatrix} (x^2) & x_0^2 & x_1^2 & x_2^2 \\ (\lambda x) & \lambda_0 x_0 & \lambda_1 x_1 & \lambda_2 x_2 \\ (ax) & a_0 x_0 & a_1 x_1 & a_2 x_2 \\ (x) & x_0 & x_1 & x_2 \end{vmatrix} = 0,$$

which vanishes identically because the first column is the sum of the others. Removing the factor $x_0 x_1 x_2$ and expanding:

$$(x^2)\,|\lambda\ a\ 1| - (\lambda x)\,|x\ a\ 1| + (ax)\,|x\ \lambda\ 1| - (x)\,|x\ \lambda\ a| = 0.$$

Substituting $(x)^2-2(x_1x_2)$ for (x^2), applying (L) to $|x\ \lambda\ a|$ and clearing of fractions, we obtain, in the abridged notation,

$$(2s_{001}^2-6s_{002})\,a_{110}+(s_{100}\,s_{001}-3s_{101})\,a_{011}+(3s_{011}-s_{010}\,s_{001})\,a_{101}=0.$$

Section (h): Geometric Interpretation.

In the following we shall denote $\sqrt{\lambda_i}$ by l_i, and $\sqrt{a_i}$ by b_i. Then, it will be remembered, l is the length of a side of the reference triangle, and b_0, $\pm b_1$, $\pm b_2$, are the coordinates of four points having the reference triangle as their diagonal triangle. So far, we have considered the a's, λ's and l's merely as magnitudes, but, of course, they are also the coordinates of certain points, the positions of which it is well to establish. A point h_0, h_1, h_2, will be denoted by the symbol h.

It is known that

1 is the centroid of the triangle,

l is the incenter,

λ is the symmedian point—the center of perspective of the vertices of the triangle with the intersections of tangents to the circumcircle at the vertices.

$1/\lambda$: Rays from a vertex through λ and $1/\lambda$ meet the opposite side in points equidistant from its mid-point.

To locate a:

Given points b_0, $\pm b_1$, $\pm b_2$, the lines joining them, two at a time, are on the vertices of the triangle—two lines on each vertex—and form a harmonic pencil with the two sides of the triangle on the same vertex. Two such lines, on 1, 0, 0, are

$$b_2x_1-b_1x_2=0, \text{ and } b_2x_1+b_1x_2=0,$$

which, taken together, form a degenerate conic

$$b_2^2x_1^2-b_1^2x_2^2=a_2x_1^2-a_1x_2^2=0.$$

The polar line of this with respect to 1 is

$$a_2x_1-a_1x_2=0,$$

on which lies a.

From the nature of this construction, the line on 1, 0, 0 and 1, (the median) is the harmonic conjugate of $a_2x_1-a_1x_2=0$ with respect to the two lines of the degenerate conic.

Hence, given a point b, to construct point a:

Join b to a vertex of the reference triangle by a line m and take n, the fourth harmonic of m with respect to the sides of the triangle on that vertex.

Take the fourth harmonic p of the median with respect to m and n. Performing this construction for each vertex, the three lines p will meet in a.

λ^2: By the above rule λ^2 is obtainable from λ.

It is interesting to note that, if b_0, $\pm b_1$, $\pm b_2$ are the coordinates of four *lines*, the *line* a is on the mid-points of the diagonals of the four-line configuration.

λa: Taking the polar of the degenerate conic $a_2x_1^2-a_1x_2^2=0$, with respect to $1/\lambda$, we have

$$\lambda_2a_2x_1-\lambda_1a_1x_2=0,$$

on which lies λa. The details of the construction readily follow.

If we calculate the polar systems of $x_0x_1x_2=0$, (the triangle itself) and $|x^2\ x\ 1|=(x_1-x_0)(x_1-x_2)(x_2-x_0)=0$ (the three medians) with respect to the various points defined above, it is seen that they are closely identified with the complete system of Section (f). In several cases, the forms of the complete system and of the polar system are identical, and, most generally, they are members of the same pencil. Hence, the geometrical construction of the complete system can be based upon the construction of the polar system.

The expanded form of $|x^2\ x\ 1|$ in the preceding paragraph shows it to be the discriminant of $x_0x_1x_2=0$. Similarly for $|\lambda^2\ \lambda\ 1|$ and $|a^2\ a\ 1|$.

Systems of Pencils of Lines in Ordinary Space.

By Alton L. Miller.

In his classic paper entitled "Preliminari di una Teoria delle Varietá Luoghi di Spazi," Segre* laid the foundation of investigations of the projective differential properties of geometric configurations in n-dimensions by synthetic methods. It is the purpose of this paper to apply some of the results of these investigations to the study of families of pencils of lines in ordinary space. By means of the Klein coordinates of a line, there is set up a one to one correspondence between lines of space and points of a hyperquadric, Q, in five dimensions. To a line on this hyperquadric there corresponds in S_3 a pencil of lines. Thus, to ruled varieties on Q there correspond families of pencils in S_3. This correspondence is discussed completely in the following paper.

Let $(x)=(x_0, x_1, x_2, x_3, x_4, x_5)$ be the Klein coordinates of a line in ordinary space, then

$$(xx)=\sum_0^5 x_i^2=0. \tag{1}$$

If at the same time we think of (x) as a point in five dimensions, equation (1) represents a hyperquadric, Q, in that space. Thus, to every line in ordinary space there corresponds in five dimensions a point on the hyperquadric, Q.

If two lines intersect, their coordinates satisfy the relation $(yz)=\sum_0^5 z_i y_i=0$, and the coordinates of any line in their pencil are given by $(x)=\lambda(y)+\mu(z)$ for some value of $\lambda:\mu$. The points of Q which correspond to two intersecting lines are therefore conjugate with respect to Q and every point $(x)=\lambda(y)+\mu(z)$ lies on Q, on the line from (y) to (z).

Part I.

1. A ruled surface, R, on Q corresponds in S_3 to a one-parameter family of pencils of lines, **R**. The centers of these pencils, in general, trace a curve C, while their planes, in general, envelop a developable, c. If R is developable,

* *Rendiconti del Circolo Matematico di Palermo*, Vol. XXX (1910).

C is the edge of regression of c. Points on the edge of regression of R correspond to the characteristics of c. If R lies in an S_4 the corresponding **R** lies in a linear complex, and if R lies in an S_3 **R** belongs to a linear congruence. This congruence will have distinct, coplanar, or coincident directrices according as R lies in an S_3 which cuts Q in a non-degenerate quadric, a pair of planes, or a cone.

PART II.

2. A two-parameter family of lines on Q generates a ruled V_3 which we will call M. To M there corresponds in S_3 a two-parameter family of pencils generating a line complex which we will call an m-complex. Every linear complex is an m-complex. In general the centers of the ∞^2 pencils trace a surface, S, and their planes envelop a non-developable surface, s. S may reduce to a curve or even a point, and, in the dual case, s may reduce to a developable or even a plane. In general S and s do not coincide.

3. We can represent M analytically as follows: Let $x_i=\alpha_i(u, v)$ and $x_i=\beta_i(u, v)$ $i=0, 1, \ldots, 5$ be any two surfaces on Q. That is $(\alpha\alpha)=(\beta\beta)=0$ for all values of u, v. Furthermore, let us assume that if A and B are two corresponding points of α and β, that is two points obtained by giving u and v the same values in α and β, then the line from A to B lies entirely on Q. The necessary and sufficient condition for this is

$$(\alpha\beta)=0 \text{ for all values of } u \text{ and } v. \tag{2}$$

Then lines joining corresponding points of α and β generate a ruled V_3 of Q, M. Hence M is given by

$$x_i=\alpha_i(u, v)+t\beta_i(u, v) \qquad i=0, 1, \ldots, 5. \tag{3}$$

If we fix u and v in (3) but cause t to vary (x) traces a line generator of M.

In S_3 the α and β surfaces represent line congruences of such a nature that corresponding lines of each intersect. If we fix u and v in (3), but cause t to vary, (x) traces a pencil of lines, one of the generating pencils of the m-complex represented by (3).

The tangent S_3 to M at (x_0), *i. e.*, $u=u_0$, $v=v_0$, $t=t_0$, is determined by (x), (x_u), (x_v), (x_t), that is by the four points

$$(\alpha), \quad (\beta), \quad (\alpha_u+t\beta_u), \quad (\alpha_v+t\beta_v). \tag{4}$$

Let G represent the generator from α to β. If we allow t to vary in (4), but hold u and v fast, (4) determines the various tangent S_3's to M at points of G.

We will call the linear congruence corresponding to the tangent S_3 to M, *tangent* to the m-complex at the line corresponding to the point of contact of

the S_3. Such a tangent linear congruence will always be degenerate in that its directrices will be coincident. It will be determined by a line of m and three lines of m infinitely near the first, but not lying in the same regulus with the first. For example by (x), $(x)+(x_u)\Delta u$, $(x)+(x_v)\Delta v$, $(x)+(x_t)\Delta t$, where Δu, Δv, Δt are all infinitesimal. Evidently the S_3 in S_5 determined by these four points is precisely the same as the S_3 determined by the points of (4), although those points do not lie on Q and therefore do not correspond to lines in ordinary space. All of the linear congruences tangent to m at lines of the pencil g have g in common, but in general, nothing else.

4. A line of M infinitely near G will join a point of the α-surface infinitely near (α) to a point of the β-surface infinitely near (β). That is a line of M infinitely near G will be determined by $(\alpha+\alpha_u\Delta u+\alpha_v\Delta v)$, $(\beta+\beta_u\Delta u+\beta_v\Delta v)$, where Δu and Δv are infinitesimal. As we vary the ratio of $\Delta u:\Delta v$ we get ∞^1 lines of M infinitely near G. G and a line of M infinitely near G determine the S_3 of the points

$$(\alpha), \quad (\beta), \quad (\alpha_u\Delta u+\alpha_v\Delta v), \quad (\beta_u\Delta u+\beta_v\Delta v). \tag{5}$$

Segre* has shown that the locus of the ∞^1 S_3's obtained by varying the ratio $\Delta u:\Delta v$ in (5) is a quadratic cone in S_5. This quadratic cone has G for a line of double points, and one other double point in each plane of Q that contains G exists for the quartic of intersection of Q with this cone. Hence the intersection of Q with the locus of all S_3's determined by G and the lines of M infinitely near G, is a quartic with double points along G and one in each of the planes of Q that contain G.

The above theorem stated in terms of line geometry becomes:

The locus of the ∞^1 linear congruences determined by a pencil g of an m-complex, and the pencils of the m-complex infinitely near g is a quadratic complex with double lines, the lines of g and one other in the plane that contains g and one other through the center of g. This is a tetrahedral complex of the type [(22)(11)].†

5. If two of the tangent S_3's at points of a generator G have a plane in common, all the S_3's tangent to M at points of G lie in an S_4. For the S_3 tangent to M at P_1 of G is the S_3 of (α), (β), $(\alpha_u+t_1\beta_u)$, $(\alpha_v+t_1\beta_v)$, and that at P_2 of (α), (β), $(\alpha_u+t_2\beta_u)$, $(\alpha_v+t_2\beta_v)$. Since these two have a plane in common, by Grassman's theorem, they lie in an S_4, and any linear combina-

* Segre, *loc. cit.*, No. 12.

† A. Weiler, *Math. Annalen*, Vol. VII (1874). Sturm, "Liniengeometrie," III, p. 436.

tion of the above eight points lies in the same S_4. In particular the following points lie in an S_4:

$$(\alpha), \quad (\beta), \quad (\alpha_u), \quad (\beta_u), \quad (\alpha_v), \quad (\beta_v). \tag{6}$$

Hence every tangent S_3 to M at a point of G lies in this S_4. Note that it does not follow that all tangent S_3's to M at points of G have a plane in common. This is not in general true.

In line space the above theorem states: *If two of the tangent linear congruences to m at lines of a pencil g have in common besides g a second pencil h having a line in common with g, all the tangent linear congruences to m at lines of g lie in a linear complex.*

The S_3's tangent to M along a generator G all have a plane in common if, and only if, the S_3's determined by G and the lines of M infinitely near G have a plane in common. In fact the tangent S_3's of (4) are S_3's determined by G and lines of the regulus obtained when we vary t in the line from $(\alpha_u+t\beta_u)$ to $(\alpha_v+t\beta_v)$. If a plane is to be common to all these S_3's it must cut every line of this regulus. Hence the regulus cuts that plane either in a conic, which is impossible, for then G would cut the regulus twice, and all the tangents S_3's would coincide, or else in a line. Then G cuts the regulus in one point. Hence a necessary and sufficient condition that all the tangent S_3's to M along G have a plane in common is that G cut the regulus mentioned above.

But the S_3's determined by G and the lines infinitely near G are the same S_3's as those determined by G, and the lines of the regulus obtained by varying the ratio $\Delta u:\Delta v$ in the line from $(\alpha_u\Delta u+\alpha_v\Delta v)$ to $(\beta_u\Delta u+\beta_v\Delta v)$. And a necessary and sufficient condition that all these S_3's have a plane in common is that G cut this regulus. But the two reguli that determine the tangent S_3's with G and the S_3's of G and the near by lines, are conjugate reguli of the same quadric. Thus, if G cuts one regulus it cuts the other also. Hence the theorem.

In line space this theorem becomes: *The tangent linear congruences to m at lines of a pencil g have in common a second pencil h having a line in common with g if, and only if, the linear congruences determined by g and the pencils of m infinitely near g have in common a second pencil h' having a line in common with g.*

If all the S_3's determined by G, and the lines of M infinitely near G have in common a plane of Q, they all coincide. Thus, *if all the linear congruences determined by g and the pencils of m infinitely near g have in common anything besides a second pencil having a line in common with g, they coincide.*

If all the tangent S_3's to M along the generators of M have a plane in common, M is said to be developable of the first kind. We will also call developable of the first kind the corresponding m-complexes, viz., those whose tangent linear congruences along the pencils of m have in common besides g, a second pencil having a line in common with g.

6. The linear congruence determined by two pencils is that congruence which contains the two pencils, and therefore the congruence which has for directrices the line joining the centers of the two pencils and the line of intersection of the planes of the two pencils. Hence the directrices of a linear congruence determined by a pencil g and a second pencil of m infinitely near g are a tangent line to S at the center of g, and a tangent line to s at the point of contact of g.

Let us consider the nature of these surfaces S and s when m is developable of the first kind. That is all the linear congruences whose directrices are corresponding tangent lines to S and s have in common g and a second pencil h which has a line y in common with g. Let P be the center and π the plane of g. P lies on S and π is tangent to s at P'. The tangent plane to S at P is π'. Then the directrices of the linear congruences referred to above will be corresponding lines of the pencils $P\pi'$ and $\pi P'$. If all of these congruences are to contain h, every line of h must cut every line of the pencils $P\pi'$ and $\pi P'$. Since h is different from g it must then be the pencil $P'\pi'$; and y, the line joining corresponding points of S and s, is tangent to both these surfaces. Hence the lines y form a congruence whose focal surfaces are S and s.

Conversely, the two-parameter family of pencils having as centers one set of focal points of a line congruence and planes, the corresponding focal planes of the congruence is an m-complex of the first developable kind. Thus every line congruence generates two m-complexes of this kind.

7. Segre* has shown that if M is a V_3 in S_5 of the first developable kind the lines of M are all tangent to a surface or else all cut a curve. Let us consider first the case in which all the lines of M are tangent to a surface. This surface lies on Q and may be taken as the α-surface. At every point of α there are two lines tangent to α and lying on Q. β may be chosen as any surface cutting one set of these lines, for example,

$$\beta = \alpha_u + \rho\alpha_v. \tag{7}$$

But β must be a point on Q, hence

$$\rho^2(\alpha_u\alpha_u) + 2\rho(\alpha_u\alpha_v) + (\alpha_v\alpha_v) = 0. \tag{8}$$

* Segre, *loc. cit.*, No. 29.

The two values of ρ obtained by solving (8) are the two directions of lines tangent to α and lying on Q. Call them ρ_1 and ρ_2. Let G_1 and G_2 be the two corresponding lines tangent to α. The locus of lines G_1 will be M_1 and the locus of G_2 will be M_2. The points of (4) determining the tangent S_3 become

$$(\alpha),\quad (\alpha_u+\rho\alpha_v),\quad [\alpha_{uu}+2\rho\alpha_{uv}+\rho^2\alpha_{vv}+\alpha_v(\rho_u+\rho\rho_v)\alpha_u+t(\alpha_{uu}+\rho\alpha_{uv})+t\rho_u\alpha_v], \tag{9}$$

and the points of (5) determining the S_3 of G and a line of M infinitely near G become

$$(\alpha),\quad (\alpha_u),\quad (\alpha_v),\quad [(\alpha_{uu}+\rho\alpha_{uv})\Delta u+(\alpha_{uv}+\rho\alpha_{vv})\Delta v]. \tag{10}$$

Thus, when M is a developable of the first sort the tangent S_3's to M along G, and the S_3's of G and the lines of M infinitely near G form two pencils of S_3's which lie in the same S_4 of (6). The plane axis of pencil (9) (as we vary t) depends on ρ, but that of (10) does not. Thus (9) furnishes two distinct pencils (9_1) and (9_2), while (10) furnishes two pencils (10_1) and (10_2) with the same central plane.

G_1 lies in two planes on Q, π_1 and π_1' and, similarly, G_2 in π_2 and π_2'. Since π_1 and π_2' are in different systems of planes of Q they intersect in a line L, and π_2 and π_1' intersect in L'. The four pencils of (9) and (10) cut on L and L' projective ranges. Projective ranges are also set up on G_1 and G_2 by means of (9_2) and (9_1), respectively.

In S_3 the congruence which corresponds to the tangent α-surface of S_5 is the y-congruence. To M_1 and M_2 there correspond the two m-complexes m_1 and m_2 mentioned in No. 6 as generated by the y-congruence. Thus, *if the m-complex is developable of the first sort and the y-congruence is not degenerate (i. e., a ruled surface) the tangent linear congruences to m along g form a pencil of congruences, and the linear congruences whose directrices are the corresponding tangent lines to the focal surfaces of the y-congruence also form a pencil of congruences. The centers (two pencils with a common line) of the last-named pencils of congruences for m_1 and m_2 coincide, while those of the first named do not.*

The lines L and L' correspond in S_3 to the pencils of tangent lines at corresponding points of S and s. Hence the above pencils of congruences cut off projective pencils of lines on these pencils.*

The necessary and sufficient condition that g and h coincide, *i. e.*, that S and s coincide is that the roots of (8) be equal. Hence

$$(\alpha_u\alpha_v)^2-(\alpha_u\alpha_u)(\alpha_v\alpha_v)=0\,\dagger \tag{11}$$

---

* These are the projectivities of Waelsch, see "Zur Infinitesimalgeometrie der Strahlencongruenzen und Flächen," *Sitzungsbericht Akad. Wien Mathem. Classe*, i. 100 Abth. IIa (1891).

† Waelsch, *loc. cit.*, obtains the same result in a different way.

is a necessary and sufficient condition that S and s coincide. It will appear later that this is a necessary and sufficient condition that M have a fixed tangent S_3 along every generator.

Since pencils (10_1) and (10_2) have the same center, a necessary and sufficient condition that they cut off the same projectivities on L and L' is that they lie in the same S_4. A necessary and sufficient condition for this is

$$|\alpha, \quad \alpha_u, \quad \alpha_v, \quad \alpha_{uu}, \quad \alpha_{uv}, \quad \alpha_{vv}| = 0, \tag{12}$$

that is, the α-surface is a ϕ-surface of Segre.*

In line space this reads: A necessary and sufficient condition that the two pencils of linear congruences whose directrices are corresponding tangent lines of S and s should cut off the same projectivities on the pencils of lines tangent to S and s at corresponding points is that the y-congruence satisfy (12). But this says that the y-congruence is a W-congruence.†

A necessary and sufficient condition that the plane common to all the tangent S_3's to M along G should lie on Q, that is a necessary and sufficient condition that the tangent linear congruences to m at lines of a pencil g have in common a plane of lines or a bundle of lines, is that the plane of the first three points of (9) lie on Q. But this plane is the osculating plane to the curve of α, along which the lines of M are tangent. Hence the above condition becomes that the curves on α, along which the lines of M are tangent, should have their osculating planes lying on Q.

If α is a ruled surface the two M's defined are first, M_1 consists of the tangent lines to α along the rulings. Then M_1 is a two-dimensional variety, α, and not a three as we have considered. Second, M_2 consists of the other set of lines tangent to α and lying on Q. The corresponding y-congruence in S_3 is made up of ∞^1 pencils of lines, and the surfaces S and s are a curve and a developable, respectively. m_1 is a family of the sort described in Part I. If α is a developable it is a plane.

If α is cut by ∞^1 planes of Q in curves, M_1 will consist of the lines tangent to these plane curves. Conversely, if the lines of M can be grouped into the tangents to ∞^1 plane curves, these planes lie on Q. In the corresponding case in S_3 the y-congruence is made up of the lines of ∞^1 cones or the tangent lines to ∞^1 plane curves. Then either S will be a curve and s a non-specialized

* Segre, "Su una classe di superficie degli iperspazi legata colle equazione lineare alle derivate parziali di 2 ordine," *Atti della R. Accad. delle Scienze di Torino*, Vol. XLIX (1913–14), p. 215.

† Darboux, "Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal," Edition II (1889), p. 345.

surface, or s will be a developable and S a non-specialized surface. Conversely, if one of the surfaces is a curve or a developable, the surface α can be cut by ∞^1 planes of Q in curves. If both S and s are curves, or both are developables, the two sets of lines tangent to α and lying on Q will be tangent to α along plane curves.

If the lines of M, instead of being tangent to a surface, all cut a curve on Q, ∞^1 of them pass through each point of this curve. Then all the pencils of m have a line in common with a ruled surface. Hence, S and s are this surface, but S and s do not coincide in the sense that the center of the pencil of m on S is the point of contact of the pencil with s. The y-congruence also reduces to this surface.

8. It may happen in S_5 that the tangent S_3's to M along G coincide, that is, there is a fixed tangent S_3 to M along every generator. Then M is said to be developable of the second kind. If M is a developable of the second kind, the α and β surfaces of (3) satisfy two linear homogeneous partial differential equations as follows:

$$\left.\begin{array}{l} A_1\alpha_i+B_1\beta_i+C_1\alpha_{i_u}+D_1\beta_{i_u}+E_1\alpha_{i_v}+F_1\beta_{i_v}=0, \\ A_2\alpha_i+B_2\beta_i+C_2\alpha_{i_u}+D_2\beta_{i_u}+E_2\alpha_{i_v}+F_2\beta_{i_v}=0, \end{array}\right\} i=0,\,1,\,\ldots,\,5, \tag{13}$$

where $A_j, B_j, \ldots, F_j$, $j=1, 2$ are functions of u and v, and not all identically zero. Conversely, if α and β satisfy (13), M is a developable of the second sort. For under these circumstances the eight points of (6) all lie in an S_3, the tangent S_3 to M along G. Accordingly, if M is developable of the second kind, all the S_3's determined by G and the lines of M infinitely near G coincide. In fact, all the lines of M infinitely near G under these hypotheses, lie in the two planes of Q that contain G. For they all lie in the tangent S_3 to M along G which cuts Q in the two planes containing G.

On the other hand, if all the lines of M infinitely near G intersect G, M is a developable of the second kind. Thus, in S_3 if m is developable of the second kind, all the lines of pencils of m infinitely near g, lie in a special linear congruence with intersecting but distinct directrices which belong to the pencil g. Every pencil of m infinitely near g has a line in common with g and, conversely, if every pencil of m infinitely near g has a line in common with g the m-complex is developable of the second kind.

9. Segre has shown* that if M is developable of the second kind, all the lines of M are tangent to two surfaces of Q, or cut a curve of Q and are tangent to a surface, or cut two curves of Q. Since we know from No. 8 that if

* Segre, "Preliminari ....," No. 29.

m is developable of the second sort all the pencils of m infinitely near g have a line in common with g, we see that either they have their centers in the plane of g or their planes pass through the center of g. In either case S and s coincide. Hence, if m is developable of the second kind, S and s coincide. See (11). Conversely, if S and s coincide, m is developable of the second kind. By the theorem of Segre every pencil of m is tangent to two congruences (or tangent to a congruence, and has a line in common with a ruled surface, or has a line in common with two ruled surfaces). That is there exist two congruences, excluding for the moment the other two cases, such that every pencil of m has two infinitely near lines in common with each. Thus S must be the focal surfaces (coincident) for both of them, and they consist of the tangent lines to S along the two sets of asymptotic lines.

If the lines of M all cut a curve and are tangent to a surface of Q, the tangents to one set of asymptotics of S involve only ∞^1 lines and therefore S is a ruled surface. Similarly, if the lines of M all cut two curves, S has two sets of rulings and is therefore a quadric.

PART III.

10. A three-parameter family of lines on Q in S_5 may generate either a V_3 or a V_4. If they generate a V_3, that V_3 is either the intersection of Q with an S_4, or consists of ∞^1 planes imbedded on Q. We will demonstrate this in two parts. First let us assume that the V_3 is developable of the second kind. That is the tangent S_3 to V_3 along G is fixed. Through every point of V_3 there are ∞^1 lines of V_3, which lie in the tangent S_3 to V_3 at that point. But this S_3 is tangent to Q along G, and therefore cuts Q in the two planes that contain G. Hence the lines of V_3 in that S_3 lie in one or two planes of Q, and V_3 consists of ∞^1 planes of Q.

If V_3 is not developable of the second kind the tangent S_3's along G cut it in ∞^1 cones of lines which are distinct and V_3 is the locus of these cones. Through every point of V_3 there are ∞^1 lines and every line cuts ∞^2 others. Hence some line besides G in one of these cones cuts some other cone, and the two S_3's of these cones have in common besides G another point and therefore a plane. Hence all the tangent S_3's to V_3 along G lie in an S_4 according to No. 5. Hence V_3 is the intersection of Q with this S_4. Hence, in line space a three-parameter family of pencils of lines either involves all the lines of space or else ∞^3 of them. If they all lie in a complex, that complex is either a linear complex or else is made up of the lines in ∞^1 planes or through ∞^1 points.

11. In what follows let M_4 represent the locus of ∞^3 lines of Q which do not lie on a V_3. Then M_4 coincides with Q. Hence all the tangent S_4's to M_4 along a generator G have an S_3 in common. Therefore, according to Segre,* all the lines of M_4 are tangent to two V_3's, or either V_3 may be replaced by a director variety of less dimension.

The S_3's of G and the lines of M_4 infinitely near G are ∞^2 in number and may be grouped into ∞^1 pencils of S_3's whose axial planes themselves form a pencil of planes about G. A proof of this theorem follows. M_4 can be represented analytically by means of

$$x_i = \alpha_i(u, v, w) + t\beta_i(u, v, w) \qquad i=0, 1, 2, \ldots, 5, \tag{14}$$

where α and β are the two tangent or director varieties mentioned in the theorem of Segre. Therefore

$$\left.\begin{aligned} \sigma\alpha &= \lambda\beta + \mu\beta_u + \nu\beta_v + \rho\beta_w, \\ \sigma\beta &= \lambda'\alpha + \mu'\alpha_u + \nu'\alpha_v + \rho'\alpha_w. \end{aligned}\right\} \tag{15}$$

The lines of M_4 infinitely near G are the lines from

$$(\alpha + \alpha_u\Delta u + \alpha_v\Delta v + \alpha_w\Delta w) \quad \text{to} \quad (\beta + \beta_u\Delta u + \beta_v\Delta v + \beta_w\Delta w),$$

where Δu, Δv, and Δw are infinitesimal. Hence the S_3's of G and the lines of M_4 infinitely near G are the S_3's of

$$(\alpha), \quad (\beta), \quad (\alpha_u\Delta u + \alpha_v\Delta v + \alpha_w\Delta w), \quad (\beta_u\Delta u + \beta_v\Delta v + \beta_w\Delta w). \tag{16}$$

As we vary the ratios $\Delta u : \Delta v : \Delta w$ in (16) we get the ∞^2 S_3's determined by G and the lines of M_4 infinitely near G. Call the plane of (α_u), (α_v), and (α_w) the plane π_α and similarly the plane π_β is the plane of (β_u), (β_v), and (β_w). According to (15) G cuts π_α in P_α and cuts π_β in P_β. Points represented by the same ratios of Δu, Δv, and Δw are corresponding points in the planes π_α and π_β. A line in π_α corresponds to a line in π_β. Let H be a line in π_α through P_α, and H' the corresponding line in π_β. Let K be a line joining corresponding points of H and H'. Then the S_3 of G and K is one of the S_3's of (16), and it contains the plane of G and H. As K moves along H these S_3's form a pencil of S_3's about the plane GH. But H was any line of π_α through P_α. Hence these planes themselves form a pencil about the line G. Hence the theorem. Similarly the S_3's of G and the lines of M_4 infinitely near G can be grouped into ∞^1 pencils of S_3's whose axial planes are determined by G and lines in π_β through P_β, that is, the axial planes form a pencil of planes about G in the tangent S_3's to π_α and π_β.

---

* Segre, "Preliminari ....," No. 29.

In line space the preceding facts give us that: All the tangent linear complexes to m_4 at lines of a pencil g are special, with the lines of contact of the pencil g as axes, and have in common a linear congruence whose directrices belong to the pencil g. The linear congruences determined by g and the pencils of m_4 infinitely near g can be grouped into ∞^1 pencils of congruences having in common besides g a second pencil h which has a line in common with g. The locus of these pairs of pencils is a special linear congruence whose directrices are coincident. This last congruence is tangent to α or β at the line which g has in common with α or β, where α and β are line complexes to which all the pencils of m_4 are tangent (or congruences or ruled surfaces with which every pencil of m_4 has a line in common). The m_4 may be considered as the locus of ∞^1 m's of the first developable kind as in Part II.

The ordering of M_4 into ∞^1 m's of the first developable kind can be accomplished in only two ways corresponding to the two ways of ordering the S_3's of G, and the lines of M_4 infinitely near G into ∞^1 pencils of S_3's. For let $f(u, v, w)=0$ be any relation on the parameters u, v, w which yields a developable V_3 of the first kind on M_4. Then

$$f_u\Delta u+f_v\Delta v+f_w\Delta w=0, \tag{17}$$

and (17) represents a line H in π_α, and a corresponding line H' in π_β. Lines joining corresponding points of H and H' form a regulus, and the S_3's of G and the lines of this regulus are the S_3's of G and the lines of M_4 infinitely near G, and lying on the V_3, determined by $f=0$. If this V_3 is to be developable of the first kind all these S_3's must have a plane in common and G must cut this regulus. Let us assume that this intersection takes place outside of π_α and π_β. Then the line of the regulus which cuts G cuts π_α in Q_α and π_β in Q_β. Hence P_α, P_β, Q_α, and Q_β are coplanar, and π_α, π_β, and G all lie in the same S_4. Hence the points (α), (β), (α_u), (α_v), (α_w), (β_u), (β_v), (β_w) all lie in an S_4, and the tangent S_4 to M_4, and therefore to Q along G is fixed, which is impossible. Hence the assumption that G cuts the above regulus in a point outside of π_α and π_β is false. Hence H passes through P_α or H' passes through P_β and the two cases of the preceding paragraph are unique.

Hence the only f's which give developable V_3's of the first kind are those for which the line (17) passes through P_α or P_β. The coordinates of these points may be obtained from (15), the point $(\sigma\alpha-\lambda\beta)$ is the point of G in π_β, and therefore has for coordinates $\mu:\nu:\rho$ which is the point P_β, similarly P_α is

$\mu':\nu':\rho'$. Hence the developables of the first kind on M_4 and on m_4 are given by one of the following equations:

$$\left.\begin{aligned} \mu f_u + \nu f_v + \rho f_w &= 0, \\ \mu' f_u + \nu' f_v + \rho' f_w &= 0. \end{aligned}\right\} \quad (18)$$

If α or β or both are surfaces instead of V_3's, there will exist a relation on the three parameters of that variety and one of them can be eliminated. Hence either μ, ν or ρ is zero, or one of μ', ν', ρ' is zero, or both. This will not affect the above reasoning. If either α or β is a curve, the pencils of S_3's determined by G and the lines of the regulus from H to H' collapse into single S_3's, for then either H or H' is a point. That is, the M_4 can be grouped into ∞^1 M's of the second developable kind.

Thus M_4's may be classified according to the type of differential equations (15) that they satisfy. If all the coefficients μ, ν, ρ and μ', ν', ρ' are different from zero, M_4 is made up of lines tangent to two V_3's. If one of these coefficients is zero, the lines of M_4 are tangent to a V_3 and cut a surface. If one of the primed and one of the unprimed coefficients are zero, the lines of M_4 cut two surfaces. If two of the primed or two of the unprimed coefficients are zero, the lines of M_4 are tangent to a V_3 and cut a curve. Other cases can not occur.

Similarly m_4's may be grouped according to the type of differential equations (15) the α and β complexes satisfy. If none of these coefficients is zero, the pencils of m_4 are tangent to two complexes. If one is zero they are tangent to a complex and have a line in common with a congruence. If two are zero in different equations they have a line in common with two congruences. If two are zero in the same equation they are tangent to a complex and have a line in common with a ruled surface. In the last case the pencils can be grouped into ∞^1 m's of the second developable type.

12. If the lines of M_4 can be grouped into ∞^1 M's of the second developable type, there must exist an $f(u, v, w)=0$ for which the tangent S_3 along G is fixed, and therefore, lies in both the tangent S_4 at α, viz., $(\alpha x)=0$, and the tangent S_4 at β, viz., $(\beta x)=0$. Hence the last two points of (16) must lie in the S_3 of intersection of $(\alpha x)=0$ with $(\beta x)=0$ for all values of $\Delta u:\Delta v:\Delta w$ which satisfy (17). These four conditions reduce to

$$(\alpha\beta_u)\Delta u+(\alpha\beta_v)\Delta v+(\alpha\beta_w)\Delta w=0 \text{ by virtue of (17).}$$

Hence

$$(\alpha\beta_u):(\alpha\beta_v):(\alpha\beta_w)=f_u:f_v:f_w. \quad (19)$$

And (19) is a necessary and sufficient condition that the lines of M_4 be susceptible to an arrangement into ∞^1 M's of the second developable kind. Furthermore (19) gives us the f's that are developable.

Hence (19) is a necessary and sufficient condition that the pencils of m_4 be capable of grouping into ∞^1 m's of the second developable type, that is whose focal surfaces, S and s, coincide, or such that the locus of the centers of the pencils and the envelope of their planes coincide.

The Pfaffian differential equation

$$X(x, y, z)\,dx+Y(x, y, z)\,dy+Z(x, y, z)\,dz=0 \tag{20}$$

defines precisely a three-parameter family of pencils of lines in ordinary space. (19) is the condition of integrability. The preceding work shows how to pick out in Klein coordinates the two-parameter families of pencils of lines for which the centers of the pencils lie on a surface S, and their planes are tangent to a surface s, where S and s are of such a nature that lines joining corresponding points of each are tangent to both in the non-integrable case. Voss * has discussed three-parameter families of pencils from this point of view.

UNIVERSITY OF MICHIGAN, Ann Arbor, Mich., *March* 17, 1917.

* Voss, "Zur Theorie den allgemeinen Punktebenesysteme," *Math. Annalen*, Vol. XXIII, p. 45.

Some Contributions to the Geometry of Plane Transformations.

BY TOBIAS DANTZIG.

§ 1. *The Indicatrix of a Transformation.*

Let T be a continuous plane point-to-point transformation sending a plane Π into itself. To fix the idea I shall further assume that T transforms every point P in the plane Π into a unique point $\overline{P}$. The latter I shall call the image of P by T. In the concluding pages of this paper is to be found an extension of most of the considerations here developed to the case of a $p:q$ transformation.

Let P be a point of the plane Π and let $\overline{P}$ be its image by T. Consider an arbitrary curve C passing through P and let t be its tangent at the point P. The curve C is transformed by T into a curve $\overline{C}$ which will pass through $\overline{P}$ and have there a unique tangent $\bar{t}$, providing the point P is a simple point of the curve C. Let the two lines t and $\bar{t}$ meet in a point τ. Then it is easy to see that,

The position of the point τ is independent of the curve C and is perfectly determined once the points P and $\overline{P}$ and the line t are given.

Indeed let C' be any other curve passing through P and having there t for tangent. The image $\overline{C}'$ will pass through $\overline{P}$ and, *since T conserves contact*, will touch at $\overline{P}$ the line $\bar{t}$. The point τ I shall call the *tactal* of the line t at the point P.

Assuming now the points P and $\overline{P}$ fixed, to each line t through P will correspond a unique line $\bar{t}$ through $\overline{P}$. When the line t turns about P a *one-to-one correspondence* is established between the two pencils having their vertices in P and $\overline{P}$, respectively, and the point is the *product* of two corresponding rays. It follows at once, therefore, that

The locus of the tactal τ for one and the same point P, is a conic K passing through P and $\overline{P}$.

The conic K is uniquely determined for every point P of the plane Π. I shall call it the *indicatrix of the transformation T for the point P.* The totality of all the indicatrices of the plane form a system of conics depending on two parameters.

When the points P and $\overline{P}$ are known, the determination of K requires the knowledge of *three* tactals. These can be obtained by tracing through P three arbitrary curves having different tangents at P. The proper selection of these curves will facilitate the construction of K. The following obvious remarks may help to guide in this selection.

The line $P\overline{P}$ I shall call the *bridge* of the transformation at P. It follows at once that

If a curve C touches the bridge $P\overline{P}$ at P, its image $\overline{C}$ will touch at $\overline{P}$ the indicatrix K, and, conversely, if C touches K at P, $\overline{C}$ will touch $P\overline{P}$ at $\overline{P}$.

If C and $\overline{C}$ have parallel tangents at corresponding points P and $\overline{P}$, the direction of these tangents is an asymptotic direction of the indicatrix K for P.

The importance of the introduction of the indicatrix can be best illustrated by the following problem:

Determine a geometrical construction of the tangent $\overline{t}$ at a point $\overline{P}$ of a plane curve $\overline{C}$, knowing that $\overline{C}$ is the image of a curve C by a transformation T.

Assume that the geometrical construction of the tangent t of the curve C is known, and let K be the indicatrix of T for the corresponding point P. If t meets K in a second point τ, τ is evidently the tactal of t at P, and consequently $\tau\overline{P}$ is the tangent sought. It is interesting to remark that if K is determined by five points (P, $\overline{P}$ and three tactals τ_1, τ_2, τ_3) and t is known, the *construction can be completed by projective means only* (Pascal's theorem).

These considerations bring out the following property: The indicatrix of a transformation T for a point P can serve as a *first asymptotic element of T at P, characterizing the behavior of T in the vicinity of the point P,* to the same degree as the tangent at a point of a plane curve characterizes the behavior of the curve at the point.

§ 2. *Equation of Indicatrix.*

To obtain the general equation of the indicatrix assume that the equations of the transformations T are given in the explicit form

$$\overline{x}=\phi(x, y), \quad \overline{y}=\psi(x, y), \tag{1}$$

where ϕ and ψ are single-valued continuous functions of x and y, admitting first derivatives which are also single valued and continuous; and where x, y;

$\bar{x}$, $\bar{y}$ are the coordinates of P and $\bar{P}$, respectively, in a system of cartesian coordinates. We shall set

$$\frac{\partial\phi}{\partial x}=p, \quad \frac{\partial\phi}{\partial y}=q, \quad \frac{\partial\psi}{\partial x}=p', \quad \frac{\partial\psi}{\partial y}=q'.$$

The equations of the corresponding tangents t and $\bar{t}$ at P and $\bar{P}$ being evidently

$$\frac{X-x}{dx}=\frac{Y-y}{dy}, \text{ and } \frac{X-\bar{x}}{d\bar{x}}=\frac{Y-\bar{y}}{d\bar{y}}, \tag{2}$$

we are to eliminate the differentials from these equations with the aid of the identical relations

$$d\bar{x}=pdx+qdy, \quad d\bar{y}=p'dx+q'dy. \tag{3}$$

We have, by setting the common ratios in (2) equal to $1/\rho$ and $1/\bar{\rho}$, respectively, the two equations

$$\bar{\rho}(X-\bar{x})=[p(X-x)+q(Y-y)]\rho, \quad \bar{\rho}(Y-\bar{y})=[p'(X-x)+q(Y-y)]\rho,$$

and eliminating ρ and $\bar{\rho}$ we obtain the equation of the indicatrix in the form

$$\frac{p(X-x)+q(Y-y)}{X-\bar{x}}=\frac{p'(X-x)+q'(Y-y)}{Y-\bar{y}}. \tag{4}$$

In this form it is apparent that the indicatrix is circumscribed to the quadrilateral whose four sides are $X=\bar{x}$, $Y=\bar{y}$, $p(X-x)+q(Y-y)=0$, $p'(X-x)+q'(Y-y)=0$. Collecting the terms we have

$$p'(X-x)(X-\bar{x})-p(X-x)(Y-\bar{y})-q'(X-\bar{x})(Y-y) \\ -q(Y-y)(Y-\bar{y})=0. \tag{5}$$

The character of the conic K is determined by the discriminant of the quadratic part of equation (5)

$$\Delta(x, y)=(p-q')^2+4p'q. \tag{6}$$

Assuming now that the transformation T is real, the curve $\Delta(x, y)=0$ divides the plane Π into two *regions: E* and H. In the region E, $\Delta<0$ and the indicatrix K is an *ellipse*, in the region H, $\Delta>0$ and the conic K is a *hyperbola*, on the *discriminant curve* itself the conic is a parabola. I shall call these regions the *elliptic* and *hyperbolic regions* respectively. If the transformation admits the whole plane for an *elliptic region* I shall call it an *elliptic transformation*, and the terms *hyperbolic* and *parabolic* transformations should be taken in the same sense.

From the consideration of the preceding section it follows at once that

At any point P in the region H there exist two rays α and β, such that, if a curve C passing through P admits one of the rays for tangent, its image $\overline{C}$ will have a parallel tangent at $\overline{P}$. These rays α and β are the asymptotic directions of the indicatrix K. They are coincident at every point of the discriminant curve Δ and become imaginary in the region E.

Of course, the locus $\Delta = 0$ is not necessarily *singly connected.* Consider for instance the *transcendental* transformation

$$\overline{x} = \cos y, \quad \overline{y} = \sin x.$$

We have here

$$\Delta = -4 \cos x \cdot \sin y,$$

and the discriminant locus consists of the lines

$$x = \pi/2 + k\pi \text{ and } y = l\pi.$$

The locus divides the plane into an *infinite number of squares* whose sides are parallel to the coordinate axes and which alternately constitute the *elliptic* and *hyperbolic regions.*

§ 3. *The Invariant Points.*

The *invariant* points of a transformation T given by equations (1) are obtained by setting

$$\overline{x} - x = \xi = 0, \quad \overline{y} - y = \eta = 0. \tag{7}$$

If equations (7) admit but a discrete number of simultaneous solutions, the invariant points are isolated. It may happen, however, that the equations $\xi = 0$, $\eta = 0$ have a factor $f(x, y)$ in common. The curve $f = 0$ is then a *curve of fixed points.*

I shall prove now that:

If S is an invariant point the indicatrix K for S degenerates into two straight lines passing through S.

Indeed, the image $\overline{C}$ of an arbitrary curve C, passing through S must also go through S. If then the tangents t and $\overline{t}$ are distinct, the tactal of t is in S, if they coincide, the tactal is indetermined on t. The coincidence of t and t will occur only when the tangent is an asymptotic direction of K. Consequently the *conic K will degenerate at an invariant point into its asymptotes.*

According as the invariant point is situated in the elliptic or hyperbolic regions, or on the discriminant curve, we shall call the invariant points *elliptic, hyperbolic, or parabolic.*

An elliptic invariant point is necessarily isolated, if real.

Indeed the indicatrix degenerates into two imaginary lines at such a point. If there were another real invariant point S', in the vicinity of S, and at an infinitesimal distance from it, SS' would necessarily be an asymptotic direction and the latter would be real, contrary to the hypothesis.

As a corollary we see at once *that in the region E no real curve can touch its image at an invariant point.*

If then T admits of a curve of fixed points the latter must be entirely in the region H or make part of the discriminant curve $\Delta=0$. In either case the curve of fixed points is the envelope of one of the branches of the indicatrix K.

Conversely, if the indicatrix of a point P degenerates into two lines passing through P, the point P is necessarily an invariant point.

EXAMPLE: Consider the transformation defined by the equations

$$\bar{x}=(x^2+y^2+2ax-a^2)/2a, \quad \bar{y}=(x^2+y^2+2ay-a^2)/2a.$$

We have in this case

$$\Delta=[(x+y)/a]^2.$$

The discriminant curve is the line $x+y=0$ counted double. The transformation is hyperbolic throughout the plane, except on Δ, where the indicatrix is a parabola. We have here

$$\xi=\eta=(x^2+y^2-a^2)/2a,$$

and the transformation admits of a fixed circle

$$x^2+y^2=a^2.$$

At any point of this circle the indicatrix degenerates into the tangent to the circle and the line parallel to $x=y$.

§ 4. *The Jacobian.*

The degeneracy of the indicatrix at an invariant point leads us to examine the general case of degeneracy of the conic K. For this purpose make in (5) the substitution

$$\xi=\bar{x}-x, \quad \eta=\bar{y}-y.$$

The equation of K then becomes

$$p'(X-x)^2-(p-q')(X-x)(Y-y)-q(Y-y)^2 \\ -(p'\xi-p\eta)(X-x)-(q'\xi-q\eta)(Y-y)=0. \quad (8)$$

The condition that the conic degenerates is

$$(pq'-qp')[p'\xi^2-(p-q')\xi\eta-q\eta^2]=0. \quad (9)$$

The degeneracy locus therefore consists of the two curves

$$J(x, y) = pq' - qp' = 0, \tag{10}$$

and

$$D(x, y) = p'\xi^2 - (p - q')\xi\eta - q\eta^2 = 0. \tag{11}$$

The first is the *Jacobian of the transformation*, the second I shall call *the degeneracy locus proper*. We see that the invariant points make part of the locus $D=0$.

Geometrically the two cases can be characterized in the following manner. If P is not an invariant point it may happen that the degenerate conic has one branch α passing through P, the other β passing through $\overline{P}$. The tactal of any line t is evidently on β; β is therefore the line corresponding to any direction t through P. The transformation therefore establishes a pseudo-correspondence between the two pencils. Such points are characterized by the fact that the image of an arbitrary curve passing through them has a fixed tangent β, independent of the direction of the tangent t. If m and $\overline{m}$ are the slopes of two curves C and C at corresponding points we have in general

$$\overline{m} = \frac{d\overline{y}}{d\overline{x}} = \frac{p' + q'm}{p + qm}, \tag{12}$$

and the condition that this homographic correspondence degenerates into a pseudo-correspondence is

$$J = 0.$$

At any point of the Jacobian of the transformation T the indicatrix degenerates into two lines of which but one α passes through P, the other β containing the point $\overline{P}$. Conversely, if the degeneracy is of this type, the point P is on the Jacobian.

It follows from these considerations that *if a curve C has in a point P belonging to the Jacobian of T a double point, its image $\overline{C}$ will have in $\overline{P}$ a cusp, and the cuspidal tangent is the branch β of the indicatrix.*

§ 5. *The Degeneracy Locus Proper.*

We shall now interpret the equation (11)

$$D = p'\xi^2 - (\gamma - q')\xi\eta - q\eta^2 = 0.$$

The equation of the indicatrix in this case is

$$p'(X-x)^2 - (\gamma - q')(X-x)(Y-y) - q(Y-y)^2 = 0, \tag{13}$$

and the condition $D=0$ expresses that one branch of the degenerate conic coincides with the bridge

$$(X-x)/\xi = (Y-y)/\eta. \tag{14}$$

At every point of the locus $D=0$ the indicatrix degenerates into two lines of which one α coincides with the bridge at P. *If the bridge is a tangent to any curve C passing through P, it is also tangent to the transformed curve $\bar{C}$ at $\bar{P}$. Conversely, if the degeneracy is of this type, the point P at which it occurs belongs to the locus $D=0$.*

If we set $\omega=\eta/\xi$ and take in consideration that

$$\frac{\partial \xi}{\partial x}=p-1, \quad \frac{\partial \xi}{\partial y}=q, \quad \frac{\partial \eta}{\partial x}=p', \quad \frac{\partial \eta}{\partial y}=q'-1,$$

equation (11) takes the remarkable form

$$\frac{\partial \omega}{\partial x}+\omega\frac{\partial \omega}{\partial y}=0. \tag{15}$$

Consider the curves, $\omega(x, y)=$const. Through any point in the plane passes a unique curve of this family. The transformation T *shifts every point of the curve, $\omega=$const., in a constant direction whose slope is ω.* The slope of the tangent at any point of the curve is equal to

$$m=-\frac{\partial \omega}{\partial x}\Big/\frac{\partial \omega}{\partial y}.$$

Relation (15) expresses that at a point where any of the curves $\omega=$const. crosses the degeneracy locus D *the curve touches the bridge of the point.*

Consider in particular a line l which the transformation T leaves invariant. At every point P of such a line the indicatrix degenerates into the line l itself and a second line which in general crosses l in a point different from P or $\bar{P}$. It follows therefore that

An invariant straight line of a transformation necessarily makes part of the degeneracy locus $D=0$. At all points of such a line the indicatrix degenerates and one branch of it is the line l itself.

From the discussion of Section 2 we conclude that *a real invariant line can not penetrate into the elliptic region of the plane*, for the degenerate indicatrix is necessarily *real* at all points of such a line.

The same remark holds for any continuous branch of the degeneracy locus and the Jacobian: *they can have in the elliptic region of the plane but isolated points.*

As a corollary it follows that an elliptic transformation admits of no invariant lines unless *a part of its discriminant locus be an invariant straight line.*

§ 6. *Directed Transformations.*

The case when the indicatrix degenerates at any point in the plane deserves special attention. Equation (9) shows at once that this will happen: *First*, when we have identically

$$\xi=0, \quad \eta=0.$$

In this case we have $\bar{x}=x$, $\bar{y}=y$, and T is the *identical transformation. Second*, when J is identically zero. There exists then a functional relation between $\bar{x}$ and $\bar{y}$, and T is a *pseudo-transformation.* Equation (1) is but a *parametrical representation* of a curve, or we can regard T as transforming the plane Π into a *one-dimensional configuration.* Leaving these cases which do not present any interest, we still have to interpret the identical vanishing of D. The general integral of equation (15) is easily found to be

$$y-\omega x=F(\omega), \tag{16}$$

where F is an arbitrary function and ω is the slope of the bridge of any point. Since, on the other hand, the equation of the bridge at a point $P(x, y)$ is

$$Y-\omega X=y-\omega x;$$

equation (16) expresses that the totality of the bridges in the plane depend on but one parameter. The bridges therefore envelope a curve W. We shall call this type of transformation a *directed transformation,* and the curve W *the directrix of T.*

If the directrix is an algebraic curve of *class k*, the multiplicity of such a transformation is at least k to k. The determination of such a transformation therefore requires the knowledge of the directrix and the *mode of correspondence* between the points P and $\overline{P}$ on the bridge. If the latter is a p-to-q correspondence, the multiplicity of the transformation is evidently $kp:kq$.

Assume, in particular, that the correspondence between the points on the bridge is *one-to-one.* All the *singular* points of the transformation are evidently situated either on the directrix or on one of the double tangents to the directrix. The transformation is *regular* at any other point of the plane, and any one of its branches may be regarded as a *one-to-one* transformation. *At any such regular point the indicatrix degenerates into the bridge and a second line β.**

* Concerning the meaning of singular and regular points see Section 21.

§ 7. *Central Transformation.*

If on a directed transformation we impose the additional condition of being of *multiplicity* 1:1, the directrix W must reduce to a *point* 0. The transformation is characterized by the fact that all bridges concur in a point, the center of the transformation.

The indicatrix of such a *central* transformation degenerates at any point in the plane into the ray of the point and a second line β.

From the preceding section it is seen that *the center is a singular point.* If we take it for origin, the equation of any central transformation can be put in the form

$$\bar{x}=g(x, y)\cdot x, \quad \bar{y}=g(x, y)\cdot y. \tag{17}$$

The curves $g(x, y)=$const. are of special interest. We have indeed,

$$\xi=(g-1)\cdot x, \quad \eta=(g-1)\cdot y, \tag{18}$$

and the locus $g=1$ is a curve of fixed points.* We find, on the other hand,

$$J=g(xg_x+yg_y+g),$$

and the *Jacobian* consists of the curves $g=0$ and $xg_x+yg_y+g=0$.

We find for the discriminant

$$\Delta=(xg_x+yg_y)^2, \tag{19}$$

and the two branches have for equation

$$yX-xY=0, \quad g_xX+g_yY=g_xx+g_yy-g(g-1). \tag{20}$$

Consequently *the second branch of the indicatrix is parallel to the tangent of the curve $g=$const. passing through the point.* At any point of the discriminant curve the curve $g=$const. touches at P the radius OP. From this it follows that *the condition that a central transformation be parabolic is that the family of curves $g=$*const. *consists of the pencil of lines through O.* Analytically the condition can be obtained by integrating (19), which gives

$$\bar{x}=g(y/x)\cdot x, \quad \bar{y}=g(y/x)\cdot y. \tag{21}$$

If we put this in polar coordinates we arrive at the conclusion that the transformation

$$\bar{\Theta}=\Theta, \quad \bar{\rho}=g(\theta)\rho \tag{21'}$$

is *parabolic throughout the plane.*

The curve $g(x, y)=k$ and its image are evidently *homothetic* with respect to the origin, the *ratio of homothety* being k.

---

* It is assumed here that if the function $g(x, y)$ has a denominator, this latter does not contain x or y as a factor.

§ 8. *The Foci of a Transformation.*

Whenever a transformation admits of an *invariant pencil* of straight lines I shall call the vertex of such a pencil *a focus* of the transformation. The following theorem brings out the importance of these points.

All indicatrices of a transformation pass through its foci, and conversely, if all the indicatrices of a transformation pass through a fixed point, the pencil of lines through this point is left invariant.

For let K be the indicatrix for any point P, $\overline{P}$ the image of P, and C a focus of the transformation. By hypothesis PC is transformed into $\overline{P}C$, C is therefore a tactal. Conversely, if all the indicatrices pass through a point C, consider any line l through C and the family of indicatrices relative to the points on l. To each point P corresponds on the conic K a point $\overline{P}$ such that $\overline{P}C$ is the tangent at $\overline{P}$ to the image of l. All the tangents of $\overline{l}$ concur therefore in C, which proves that $\overline{l}$ is a straight line passing through C.

The transformation being assumed one-to-one, there exists a *homographic correspondence* between the two *superposed pencils* l and $\overline{l}$. We see therefore, that in any invariant pencil there are *two invariant lines* u and v, the *double rays* of the pencil through C. I shall call the invariant pencil *elliptic*, *hyperbolic*, or *parabolic*, according as u and v are congugate imaginary, real, or coincident.

If an invariant pencil contains more than two invariant lines, every line in the pencil is invariant. The homographic correspondence between the two pencils l and $\overline{l}$ reduces to *identity* and the transformation is a *central* transformation.

I shall say that a transformation is *unifocal*, *bifocal*, etc., if it admits of one, two, etc., foci. It follows from the considerations developed that *among the indicatrices of a unifocal transformation there are two families of degenerate conics.* All the indicatrices of any one of the two families have an invariant line of the transformation for one common branch.

No one-to-one transformation can possess more than three non-collinear foci.

For, assume that there are four foci, C_1, C_2, C_3, C_4, no three of which are on a straight line. All indicatrices will have to pass through the four foci. The system of indicatrices in the plane would reduce to a *pencil of conics* and any one of the conics would have to serve as indicatrix for any point on it, which is absurd.

We shall see later that if a transformation admits of *three non-collinear foci* it is a *collineation.*

If three of the foci of a transformation are collinear, any point on the line δ containing these foci is also a focus.

Indeed, since any indicatrix must pass through all the foci, they will all have the *axis* δ for common branch, and, consequently, contain any point on δ. We shall see later that the transformation is a *perspective.*

§ 9. *Unifocal Transformation.*

If the focus of a unifocal transformation be taken for origin, one of the equations of the transformation can be readily obtained by expressing that there exists between y/x and $\bar{y}/\bar{x}$ a *homographic relation*

$$Ax\bar{y}+By\bar{y}=Cx\bar{x}+Dy\bar{x}. \tag{22}$$

By setting

$$\bar{x}/(Ax+By)=\bar{y}/(Cx+Dy)=g(x,y),$$

the equations of the transformation are put into the form

$$\bar{x}=(Ax+By)\cdot g(x,y),\quad \bar{y}=(Cx+Dy)\cdot g(x,y); \tag{23}$$

and this is the resultant of the two transformations

$$\bar{x}=x'g(x',y'),\ \bar{y}=y'g(x',y')\ \text{ and }\ x'=Ax+By,\ \ y'=Cx+Dy.$$

Consequently,

Any unifocal one-to-one transformation is the product of a central and a linear transformation.

Let now T_1 and T_2 be two one-to-one transformations, one sending a point P into P_1, the other sending P_1 into P_2, and let $T_1T_2=T_3$ represent the product of the two transformations. Let, moreover, K_1 be the indicatrix of T_1 for P, K_2 that of T_2 for P_1, and K_3 that of T_3 for P. *The conic K_3 must go through the three intersections of K_1 and K_2 others than P_1.* Indeed, if Q be one of these intersections, QP and QP_1 are corresponding rays in the pencils (P) and (P_1). Since, on the other hand, P_2 is on K_2, QP_1 and QP_2 are corresponding rays in the pencils (P_1) and (P_2). It follows therefore that QP and QP_2 must be corresponding rays in the pencils (P) and (P_2) as determined by the transformation T_3. Q is therefore on the conic K_3.

This lemma permits the determination of the indicatrix of a product of two transformations when the indicatrices of the components are known. For it gives five points of the conic K_3: P, P_2, Q, Q_1, Q_2. In particular it can be applied to any unifocal transformation thus *reducing the determination of the indicatrix to that of a central transformation.*

§ 10. *Bifocal Transformation.*

Consider a one-to-one transformation T admitting *two* foci, C_1 and C_2. We shall call the line $l = C_1C_2$ the *line of foci.* Each of the pencils C contains *two invariant lines.* Denote these by u_1, v_1 and u_2, v_2, respectively. It is clear that the transformation admits of *four invariant points,* u_1u_2, u_1v_2, v_1u_2, v_1v_2. The foci are singular points; the image of C_1 is indetermined on the line l_2 corresponding to l in the pencil (C_2), and similarly C_2 is transformed into the line l_1 corresponding to l in the first pencil. The transformation admits of a third singular point G, the intersection of the lines l_1' and l_2' to which l corresponds in the pencils (C_1) and (C_2), respectively. The image of G is therefore the line of foci.

On the other hand T can be considered in two ways as a *unifocal transformation.* It follows that the equations of the transformation are of the form

$$A_1\bar{x}+B_1\bar{y}+C_1=0, \quad A_2\bar{x}+B_2\bar{y}+C_2=0, \tag{24}$$

where the coefficients are linear expressions in x and y.

T is therefore a "Magnus transformation."* Any straight line is transformed into a conic going through *three fundamental points,* which in this case are C_1, C_2 and the intersection F of l_1' and l_2'. Conversely, any conic through C_1, C_2 and the intersection G of l_1' and l_2' is transformed into a straight line. In particular, the pencil of lines through G goes into the pencil through F.

Let L, M, N be the fundamental points of any Magnus transformation T, and L', M', N' the fundamental points of the *inverse* transformation T^{-1}, and consider T as operating between two superposed planes Π and Π'. Then a linear transformation could bring two pairs of corresponding points in coincidence, say L' into L and M' into M. Under these circumstances the pencils (L') and (M') would be superposed with (L) and (M), and the points L and M would be foci of the new transformation. Therefore

The most general Magnus transformation can be considered as the product of a linear transformation and a bifocal transformation.

§ 11. *Conformal Transformations, Elliptic Type.*

If a transformation conserves angles both in size and in sense of rotation I shall call it an *elliptic* transformation, otherwise a *hyperbolic conformal transformation.*

* See Magnus, "Nouvelle méthode," etc. *Crelle,* 8 pp. 52–63.

An elliptic conformal transformation has for indicatrix at any point of the plane a circle.

The theorem is evident geometrically. Indeed, let t and t' be any two rays of the pencil (P), and let $\bar{t}$ and $\bar{t}'$ be the corresponding two rays in the pencil $(\bar{P})$. By hypothesis the angles (t, t') and $(\bar{t}, \bar{t}')$ are equal and have the same sense of rotation; the locus of the point τ is therefore a circle.

Conversely, if the indicatrix is a circle, the angles are conserved. Analytically the condition that equation (8) represent a circle is given by

$$p=q', \quad q=-p', \tag{25}$$

and this is the well-known condition that a transformation be "*directly*" *conformal.*

A conformal transformation of this type is necessarily *elliptic* throughout the plane, and the discriminant curve is imaginary. *An invariant point of an elliptic conformal transformation is necessarily isolated and admits for indicatrix the isotropic lines passing through it.*

There are no real invariant lines.

It follows from the developments of the preceding section that *a conformal one-to-one transformation of this type is a bifocal transformation having two elliptic foci in the cyclic points at infinity I and J.*

All straight lines in the plane are transformed into conics going through the foci, *i. e.*, into *circles*, and these circles will have a third fixed point in common, real or imaginary. *The lines of the plane have for images circles having a radical center in common.* This radical center is *real*, for the lines l and l, to which the line at ∞ corresponds in the pencils I and J, respectively, are conjugate imaginary. For the same reason the *four invariant points of a conformal transformation are real.*

Let F be the radical center of the circles, and let G be the radical center of the circles which are transformed into lines. Then from the preceding section it is seen that F is transformed into the line at infinity and G is the image of the same line, *i. e.*, F and G are *vanishing points.*

§ 12. *Conformal Transformations, Hyperbolic Type.*

The case of the *hyperbolic conformal* transformation, in which the sense of rotation of angles is reversed, is connected with the preceding case by the following theorem:

Any hyperbolic conformal transformation is the product of a circular transformation and reflexion.

Indeed, let

$$\bar{x}=\phi(x, y), \quad \bar{y}=\psi(x, y)$$

be the equations of a circular transformation. If we make the substitution

$$x=x', \quad y=-y',$$

we will have

$$\frac{\partial \bar{x}}{\partial x'}=p, \quad \frac{\partial \bar{x}}{\partial y'}=-q, \quad \frac{\partial \bar{y}}{\partial x'}=p', \quad \frac{\partial \bar{y}}{\partial y'}=-q',$$

the condition $p=q'$, $p'=-q$ is therefore transformed into

$$p=-q', \quad p'=-q. \tag{26}$$

Nevertheless there are some advantages in treating the problem directly. The following theorem is true:

The indicatrix of a conformal transformation of the hyperbolic type is an equilateral hyperbola admitting the bridge for diameter.

The proof can be based on this classical property of an equilateral hyperbola: Any arc of the curve is viewed at equal or supplementary angles from the extremities of any diameter of the curve. It can also be regarded as the geometrical interpretation of conditions (26) in the equation of the indicatrix (8).

Let us apply these considerations to the determination of all *central conformal transformations*. It is clear that, if real, these can not be of the circular type as the indicatrix degenerates. Therefore the indicatrix is a degenerate equilateral hyperbola admitting the bridge for diameter. The second branch of the indicatrix is then the *perpendicular bisector* to $P\bar{P}$. The curves $g=$const. are *orthogonal trajectories* to the bridges as well as their images, that is, *circles having O for center*. The transformation is necessarily *reversible*, and the points of any bridge and their images determine an *involution*. The transformation is therefore an *inversion*.

§ 13. *Collineations.*

Assume now that a one-to-one transformation T admits *three invariant pencils* (A), (B), (C). *The indicatrices of the transformation form then a system of conics circumscribed to a fixed triangle ABC.*

Consider a line l of general position, and consider the family of indicatrices relative to the points on this line. They depend on but one parameter and consequently admit of an envelope (E).

Let E be the point in which the conic K relative to a point P touches the envelope. This point is uniquely determined, for by hypothesis *the indicatrix passes through three fixed points.* Let K' be the position of the indicatrix for an infinitely close point P'. The tactal τ of the line l for P is at the intersection of K and l, and likewise that of P' is at the intersection of K' and l. When the point P' approaches P, the point τ approaches the point E as a limit. The point E is therefore on l. Since all indicatrices relative to the line l can not touch the line l it follows therefore that the point E is fixed.

Consequently

Whenever a transformation admits three foci the family of indicatrices relative to the points on any straight line form a pencil of conics.

This theorem leads at once to the following:

Any trifocal transformation is a collineation.

Indeed the image of the line l has all its tangents going through the point E, it is therefore a line through E. We can complete then the preceding theorem by this statement. The four basic points of the pencil of conics are the three foci and the intersection of the line l and its image $\bar{l}$.

Among the indicatrices of the pencil, that relative to E is *tangent to the line l.* For let $\bar{E}$ be the image of E: the line $E\bar{E}$ is the bridge for E, consequently the line l must touch the indicatrix at l. (See Section 1.)

Conversely any collineation admits three foci. To show this I shall prove the following lemma:

The indicatrix of the image of a point P is the image of the indicatrix of the point P.

Indeed, let P and $\bar{P}$ be the two points. By hypothesis *any line l through P is transformed into a line $\bar{l}$ through $\bar{P}$.* Let l and $\bar{l}$ meet in a point τ. This point belongs to the indicatrix K of P. Let us seek the image $\bar{\tau}$ of τ. It is at the intersection of $\bar{l}$ and its image $\bar{\bar{l}}$, and consequently belongs to the indicatrix $\bar{K}$ of $\bar{P}$.

Assume now that the indicatrix K does not degenerate identically, that is that the collineation is *not central.* The two conics K and $\bar{K}$ meet besides in $\bar{P}$ in three other points A, B and C. The line PA goes into $\bar{P}A$, and $\bar{P}A$ into $\bar{\bar{P}}A$. *A is therefore an invariant point* and the same is evidently true of B and C. The three pencils (A), (B), (C) are therefore invariant. Consequently *any collineation is a trifocal transformation, and the indicatrices circumscribe a fixed triangle ABC.*

§ 14. *Perspective.*

In the proof of the last theorem we have expressly assumed that the indicatrix does not degenerate identically. Suppose now that the conic K degenerates at any point of the plane and let P and $\overline{P}$ be a point and its image. We know that the transformation is central, *i. e.*, $P\overline{P}$ goes through a fixed point P. Let β be the second branch of K. Any line l through P will intersect its image $\overline{l}$ in a point τ on β. It follows that if on the line l we take any other point Q the indicatrix of Q will also go through τ. And since τ is any point on β we conclude that

All indicatrices of a central collineation (*or perspective*) *have one branch in common* (*the axis of perspective*).

Any point on the axis β is a focus and the transformation possesses also an additional focus O without the axis. Let C be a point of the axis. The pencil (C) has for the two invariant lines CO and β.

The product of two perspectives is in general a non-central collineation. Indeed, let T_1 and T_2 be two perspectives with O_1 and O_2 as centers, and β_1 and β_2 as axes, respectively, and let P, P_1 and P_2 be a point, its image by T_1, and its image by T_1T_2, respectively. PP_1 goes through O_1, and P_1P_2 through O_2. According to a lemma proved in Section 9 the indicatrix of P by T_1T_2 is determined by P, $\overline{P}$ and the three intersections of the indicatrices of P and P_1 for T_1 and T_2, respectively. If then β_1 and β_2 meet in a point A, PP_1 and β_2 in F, and P_1P_2 and β_1 in G, the indicatrix of the resultant transformation will go through P, P_2, A, F and G, and consequently does not degenerate unless β_1 and β_2 coincide.

The perspectives of a plane do not form a group. All perspectives having a common axis form a group.

It is readily seen that the point A where the two axes of perspective meet is one of the foci of the resultant collineation. To find the other two foci we shall remark that T_1T_2 shifts O_1 and O_2 on the line O_1O_2. The line O_1O_2 is therefore invariant under T_1T_2 and contains the other two foci. The latter can therefore be determined as the intersections of the indicatrix for P and O_1O_2.

Conversely any collineation T can be regarded in an infinite number of ways as a product of two perspectives, T_1T_2. We can choose arbitrarily the two centers O_1 and O_2 on one of the sides of the invariant triangle, the two axes of perspective are then perfectly determined. They meet at the opposite vertex of the triangle. To actually perform the decomposition let P and $\overline{P}$ be

two corresponding points and let K be the indicatrix for P. O_1 and O_2 being taken at random on BC, let PO_1 meet K in a second point F, and $\overline{P}O_2$ meet K in G. Then FA and GA are the two axes of perspective. If now P_2 be the intersection of O_1P with $O_2\overline{P}$, P_1 is the image of P by T_1 while $\overline{P}$ is the image of P_1 by T_2. The two perspectives are therefore perfectly determined.

This theorem gives a first method of constructing a collineation given the invariant triangle and a pair of corresponding points.

§ 15. *Classification of Collineations.*

The indicatrix offers a natural method for the *classification* of *collineations.* Since the indicatrices of a collineation form a *bundle of conics* the following cases are possible:

A. *The three foci are distinct.* The indicatrices circumscribe the invariant triangle. We shall call such a collineation *trifocal.* From the standpoint of real transformations we must distinguish two cases: a) the three foci are all *real,* b) two of the foci are *conjugate imaginary.* In the first case all three invariant pencils are of the *hyperbolic* type. In the second case two of the pencils are *elliptic.*

B. *Two of the foci are coincident.* The indicatrices all pass through a fixed point A and touch a fixed line a at another fixed point B. The invariant pencil (B) is *hyperbolic* while (A) is *parabolic.* For any point of the line a the indicatrix degenerates into a and a line β passing through A; while on the other hand the indicatrix of any point on AB will have for the two branches AB and a line through B. In particular at B the indicatrix will consist of AB doubled. Therefore the point A belongs to the discriminant curve. We shall call this type a *bifocal* collineation.

C. All three foci are coincident in a point A. The collineation is *unifocal.* All indicatrices *osculate* at A and therefore have a common tangent a. The pencil (A) is parabolic and the point A again belongs to the discriminant curve.

D. One of the invariant pencils consists of invariant lines only. Suppose (C) to be this pencil; then C is a center and the opposite side c of the invariant triangle is a fixed line. The transformation is therefore a *perspective* having C for *center* and c for *axis.*

E. A particular case of the latter is when the center C is on c. In this case the transformation can be considered as a *degeneracy of the type C.* The curve c is a part of the discriminant curve. The collineation is an *elation.*

§ 16. *The Discriminant of a Collineation.*

The discriminant curve of a trifocal collineation is a parabola inscribed in the invariant triangle.

Consider indeed any line l and its image $\bar{l}$ and let E be their intersection. We have proved that the family of indicatrices relative to l is a pencil of conics having E for a *fourth fundamental point.* But among the conics of a pencil there are two parabolae: let D and D' be the points for which these parabolae are indicatrices. Then D and D' are the unique points in which l meets the discriminant curve Δ. The latter is therefore a conic.

When a point moves along any invariant side, say a, the second branch of its indicatrix turns about A; the correspondence thus generated between the range (a) and the pencil (A) is one-to-one. There exists therefore but one point L on a whose indicatrix degenerates into two parallel lines, *i. e.*, a and the line a' parallel to a and passing through A. This point L belongs to Δ and evidently a touches Δ at L. Since there is but one such point on any side of the invariant triangle, the conic Δ is *inscribed* in the triangle ABC.

On the other hand, the pencil of indicatrices relative to a line l generate a *parabolic* involution on l, since l goes through one of the basic points of the pencil and E is the double point. There exists therefore in a pencil but one conic K touching l, and the point of contact is in E. If we take for l the line at infinity, $\bar{\omega}$, the unique conic of the pencil relative to ω that touches $\bar{\omega}$ is evidently a parabola, and the point of contact belongs to the discriminant curve. The two points D and D' in which $\bar{\omega}$ meets Δ are coincident in E, and therefore D touches $\bar{\omega}$ at E. *The discriminant curve is thus a parabola.*

The asymptotic directions of the indicatrix at a point are the two tangents drawn from P to the discriminant conic Δ.

Indeed, let l be a line through P parallel to its image $\bar{l}$. The point E is then at infinity, and the conics K of the pencil touching l at E are parabolas. The pencil has therefore two coincident parabolas and consequently l is tangent to the discriminant curve Δ.

As a corollary we see that *it is the interior of the parabola Δ that makes up the elliptic region of the plane.* If then the three foci are all real (hyperbolic) the invariant triangle is entirely within the region H. If two of the foci are conjugate imaginary, the real focus, being elliptic, must be within the parabola.

As a second obvious corollary we derive that *the bridges relative to the vanishing line ω envelope the discriminant curve.* It follows from this that the

vanishing line itself *is a tangent to* Δ, for there exists on it one point $\vee$ which has for image the point at infinity on ω, and since the image of the bridge of a point touches the indicatrix at the image of the point, $\vee$ has for indicatrix a parabola and is therefore on Δ.

The same considerations hold with but slight modifications in the case of *bifocal and unifocal* collineation. If A and B are the two foci, and a and c the two invariant lines, the curve will be *inscribed in the angle* (a, c) touching c at A. In the case of a *unifocal* collineation the parabola Δ is osculating at A all indicatrices, it is therefore itself an indicatrix.

In a perspective the conic Δ *degenerates in a double line parallel to the axis of perspective and going through the center. This line coincides with the axis of perspective in the case of elation.*

§ 17. *The Conformal Points of a Collineation.*

Let l and l' be any two lines through a point P, and $\bar{l}$ and $\bar{l}'$ their images through $\bar{P}$. If the angles (l, l') and $(\bar{l}, \bar{l}')$ are equal for all positions of the lines, I shall call P a *conformal point.* I shall prove now that

There exists for any non-conformal collineation one and but one conformal point in the elliptic region. This point is the focus of the discriminating parabola.

Indeed, to determine the point it is sufficient to find the point E which has for indicatrix *the circle circumscribed to the invariant triangle*, and providing the collineation is not conformal, there exists but one such point. Now the asymptotic directions at E are *isotropic*, and since they are, according to the preceding section tangents to Δ, E is the point from which we can draw two isotropic tangents to Δ, that is, the *focus* of Δ.

In the same way it can be shown that in the *hyperbolic* region of the plane there also exists a point H at which angles are conserved. The indicatrix for this point must be an equilateral hyperbola and $H\bar{H}$ a diameter of the indicatrix. Now, since the asymptotic directions of all indicatrices are tangent to Δ, the locus of the point for which the indicatrix is a rectangular hyperbola is the *directrix d of the discriminating parabola.* The line d goes through the *orthocenter* of the invariant triangle and meets there its image $\bar{d}$, for all *equilateral hyperbola circumscribed to a triangle pass through its orthocenter. The point H may be any point on d, but once selected, the collineation is perfectly determined.* Indeed the indicatrix is then known by five points (H, orthocenter, and A, B, C) and consequently by taking for $\bar{H}$ the point diamet-

rically opposed to H on the indicatrix, the indicatrix is determined by the three invariant points and a pair of corresponding points.

The actual construction of the point $\bar{H}$ can be effected in the following manner: Let O be the orthocenter of the triangle. Then OA must be seen from the same angle from H and $\bar{H}$. If then we construct on OA *an arc of capacity* OHA the point $\bar{H}$ is on the arc, providing the arc has been so drawn that the angles $O\bar{H}A$ and OHA have opposite senses of rotation. If we now perform the same construction for OB, the point $\bar{H}$ will be determined as one of the intersections of the two circles, the other being in O.

If a collineation admits more than one conformal point of each type it is a conformal transformation. This is clear for the elliptic points, for were there more than one such point, the corresponding indicatrices being circles, two of the invariant points would be the cyclic points at infinity, and consequently all indicatrices would be circles. As to the hyperbolic conformal points the case can be immediately reduced to the preceding one by a reflection. *A non-central hyperbolic conformal collineation can not exist however, for all conics circumscribing a triangle can not be equilateral hyperbolae.*

§ 18. *Geometrical Determination of a Collineation.*

LEMMA. *If P, $\bar{P}$ and Q, $\bar{Q}$ are two pairs of corresponding points, and if the indicatrices of the collineation for P and Q meet in a fourth point S, then will PQ and $\bar{P}\bar{Q}$ meet in S.*

PROBLEM 1. *Determine a collineation by its invariant triangle and two pairs of corresponding points P, $\bar{P}$ on a and Q, $\bar{Q}$ on b.*

The second branch of the indicatrix for P passes through A, and that of Q through B, and according to the lemma both indicatrices contain the point S in which PQ and $\bar{P}\bar{Q}$ meet; the indicatrices are therefore perfectly determined. In order now to obtain the image of any point R in the plane, join R to P and let RP meet SA in R_a, also draw RQ meeting SB in R_b; then will $\bar{P}R_a$ and $\bar{Q}R_b$ meet in the image $\bar{R}$.

PROBLEM 2. *Determine a collineation by its invariant triangle and one pair of corresponding points R and $\bar{R}$ not belonging to the invariant triangle.*

We can immediately reduce this to Problem 1 by drawing RA, $\bar{R}A$, RB, $\bar{R}B$; for if the first two lines meet a in P and $\bar{P}$, and the second two meet b in Q and $\bar{Q}$, P and $\bar{P}$, Q and $\bar{Q}$ are evidently corresponding points.

The problem can also be treated directly, for the indicatrix K_R of R is determined by the five points R, $\bar{R}$, A, B and C. We can therefore determine projectively the intersection of any line RS with K. Let this intersection be τ; then the indicatrix K_S of S is perfectly determined by the five points τ, S, A, B, C, and the determination of $\bar{S}$ requires the finding of the intersection of $\bar{R}$ with K_S. The construction therefore necessitates two Pascal constructions.*

PROBLEM 3. *Determine a collineation by four pairs of corresponding points of general position:* $P, \bar{P};\ Q, \bar{Q};\ R, \bar{R};\ S, \bar{S}$.

If we denote by $PQ \cdot \bar{P}\bar{Q}$ the intersection of PQ and $\bar{P}\bar{Q}$, the conics K_P and K_Q are determined respectively by the five points

$P;\ \bar{P};\ PQ \cdot \bar{P}\bar{Q};\ PR \cdot \bar{P}\bar{R};\ PS \cdot \bar{P}\bar{S}$ and $Q;\ \bar{Q};\ QP \cdot \bar{Q}\bar{P};\ QR \cdot \bar{Q}\bar{R};\ QS \cdot \bar{Q}\bar{S}$.

By one Pascal construction we can therefore determine the intersection τ_p of K_P with the line MP, M being any point in the plane. A second Pascal construction would give us the point τ_q where MQ meets K_Q. The point $\bar{M}$ is at the intersection of $\bar{P}\tau_p$ and $\bar{Q}\tau_q$.

We have the following theorem as a consequence of the fact that *four pairs of points determine a collineation.*

Let $P, \bar{P};\ Q, \bar{Q};\ R, \bar{R};\ S, \bar{S}$ *be four pairs of points of general position. The four conics* K_P, K_Q, K_R *and* K_S *determined respectively by*

$$P, \bar{P}, PQ \cdot \bar{P}\bar{Q}, PR \cdot \bar{P}\bar{R}, PS \cdot \bar{P}\bar{S};\ Q, \bar{Q}, QP \cdot \bar{Q}\bar{P}, QR \cdot \bar{Q}\bar{R}, QS \cdot \bar{Q}\bar{S};$$
$$R, \bar{R}, RP \cdot \bar{R}\bar{P}, RQ \cdot \bar{R}\bar{Q}, RS \cdot \bar{R}\bar{S};\ S, \bar{S}, SP \cdot \bar{S}\bar{P}, SQ \cdot \bar{S}\bar{Q}, SR \cdot \bar{S}\bar{R}$$

have three points in common.

PROBLEM 4. *A collineation being given by its invariant triangle and a pair of corresponding points,* P *and* $\bar{P}$, *determines its discriminating parabola* Δ *and the elliptic conformal point.*

Applying again our lemma let the parallel to a through A meet the indicatrix K_P in S, then if PS meets a in L, L is the point where a touches Δ. The points M and N are determined in a similar manner.

The *elliptic conformal point* E is located by the following simple construction. Let the parallel through A meet the circumscribing circle K_E in τ, and let τL meet the circle in a second point E. The latter is evidently the *conformal point.*

The latter construction can be applied to the problem: *Find the focus of a parabola inscribed in a triangle.*

---

*I call Pascal construction the construction based on Pascal's hexagram theorem.

§ 19. *Displacements, Linear Transformations.*

A *conformal* collineation must be either of the *trifocal* type or a *central* collineation.

I. *Rotation.* If the indicatrices of a conformal collineation do not degenerate identically, they can not consist of equilateral hyperbolae only. They are therefore circles, and two of the invariant points are the cyclic points I and J. The transformation is a *rotation. All indicatrices have a common radical center O, the center of rotation.* If the transformation is *real* this latter is *real*, and the *bundle of circles* is a *hyperbolic* bundle. If the center be taken for origin and if α be the angle of rotation, the equation of the transformation is

$$\bar{x}=x\cos\alpha-y\sin\alpha,\quad \bar{y}=x\sin\alpha+y\cos\alpha. \tag{27}$$

The equation of the indicatrix is then

$$X^2+Y^2=k(xX+yY),\text{ where } k=\sec\alpha. \tag{28}$$

II. Still assuming that the *conformity is direct* if the indicatrix degenerates identically, this can only happen in two ways. Either the two branches are *asymptotic* directions of the circle, that is *isotropic* lines and then the transformation is imaginary; or *one of the branches is the line at infinity.* Any line l has in the latter case a *parallel* image. Assume first that the *center of perspective* is not on the *line at infinity.* The transformation is then *homothetic.*

III. If, however, the perspective becomes an *elation*, that is its center is also on the line at infinity, the transformation is a *translation.*

IV. Finally, if the conformity of the collineation is of the *hyperbolic* type, the indicatrix must consist of two perpendicular straight lines, and its *fixed branch must bisect the bridge.* The bridges are therefore all parallel and the *center of perspective is at infinity.* We have then a *reflection.*

More generally consider a collineation leaving the line at infinity invariant. If the two invariant points on the line at infinity are real, all the indicatrices are hyperbolae with *fixed asymptotic directions.* If, on the other hand, the two foci at infinity are imaginary, the transformation is elliptic throughout the plane. The collineation in both cases are *linear transformations*, and, if the finite focus be taken for origin, admit the equations

$$\bar{x}=ax+by,\quad \bar{y}=cx+dy. \tag{29}$$

The discriminant is

$$\Delta=(a-d)^2+4bc, \tag{30}$$

so that the transformation is hyperbolic or elliptic according as Δ is positive or negative. If $\Delta=0$ the two foci at infinity become coincident. The indicatrices represent a bundle of parabolae passing through a fixed point, and admitting at the point a fixed diameter. Geometrically the three cases are distinguished by the fact that a hyperbolic linear transformation will leave two real pencils of parallel lines invariant, a parabolic but one real pencil while an *elliptic linear transformation will not move any real line parallel to itself.*

The last considerations are capable of an obvious generalization. Consider any *Magnus* transformation that admits *two foci at infinity.*

The indicatrices of such a transformation will have fixed asymptotic directions, and, conversely, any one-to-one transformation admitting a system of indicatrices with fixed asymptotic directions is a Magnus transformation with two foci at infinity.

If the third fundamental point be taken for origin and the directions determined by the foci as directions of coordinate axes the equation of such a Magnus transformation takes the form

$$\bar{x}=A/(x-a), \quad \bar{y}=B/(y-b). \tag{31}$$

A and B being arbitrary coefficients, and a, b the coordinates of the finite fundamental point.

§ 20. *Dualistic Developments.*

The results reached in this investigation admit of obvious *dualistic developments.*

Given a *line-to-line transformation* T, let l and $\bar{l}$ be two corresponding lines. Consider a line configuration C admitting l as a simple tangent, and let P be the point of contact. The image $\bar{C}$ of C will touch $\bar{l}$ at a unique point $\bar{P}$. I shall continue to call $P\bar{P}$ the *bridge.* The bridge is independent of the curve C chosen and depends only on the line l and the point P. When the point P describes l there is a *one-to-one correspondence* generated between the ranges l and $\bar{l}$, and the *envelope of the bridge is consequently a conic* L touching l and $\bar{l}$. I shall call *this conic the indicatrix of the line transformation* T *for the line* l.

The discussion of the divers cases brings out the following properties of the *line indicatrix* which I state without proof:

Let S be the intersection of l and $\bar{l}$, and let L touch l and $\bar{l}$ in A and B, respectively, and furthermore let A' and B' be the points of L diametrically opposed to A and B, then

1°. *If a curve C touches l in A its image $\overline{C}$ will touch $\overline{l}$ in S, and conversely: if C touches l in S then will $\overline{C}$ touch $\overline{l}$ in B.*

2°. *If a curve C admits l for asymptotic, its image $\overline{C}$ will touch $\overline{l}$ in A', and if C touches l in B', $\overline{l}$ is an asymptotic of $\overline{C}$.*

3°. *Let C and C' be any two curves and l a common tangent, touching C in P, C' in P'. If $\overline{C}$ and $\overline{C}'$ touch $\overline{l}$ in $\overline{P}$ and $\overline{P}'$, I shall say that distances are conserved along l and $\overline{l}$ if $\overline{PP}'=PP'$.* A line transformation that conserves distances throughout the plane I shall call a *collateral transformation*, directly collateral if $\overline{PP}'=PP'$ inversely if $\overline{PP}'=-PP'$. Then *the condition necessary and sufficient that a transformation be inversely collateral is that the indicatrix be a parabola symmetrically inscribed in the angle $(l, \overline{l})$. For a direct collateral transformation it is necessary and sufficient that the indicatrix degenerate into two points, the point S where l and $\overline{l}$ meet, and the point at infinity in the direction of a bisector of $(l, \overline{l})$.*

4°. If the indicatrix for a line l degenerates, three cases are possible: A. *l is an invariant line;* B. *l is an element of the Jacobian;* C. *Any curve touching l at S will have an image touching $\overline{l}$ at S.* The degenerate indicatrix consists of two points, which in Case A are both on l; in Case B one of the points is on l, the other on $\overline{l}$; finally, in Case C, one of the points is in S.

5°. *If the indicatrix degenerates identically and T is a proper transformation the point S will describe a curve, which may again be called the directrix of T, and T a directed transformation.* If besides T is one-to-one, the directrix is necessarily a *straight line* which I shall call *the axis* of T, and T an *axial transformation.*

6°. If a range of points goes into itself the bearer of the range, f, is a *focal line.* All the indicatrices of the plane will touch f. We can prove in the same way in the correlative case that *a one-to-one transformation will not admit more than three non-concurrent focals.* In the case of three focals we have a collineation which is a *self-dualistic* transformation. If, however, three focals are concurrent in a point O, any line through O is a focal and the transformation is a *perspective.* The latter being an axial transformation, any indicatrix will consist of the point O and a second point on the axis of perspective.

7°. If a collineation T be regarded as a line transformation all *indicatrices are inscribed in the invariant triangle. The discriminating parabola of T regarded as a point collineation is nothing but the indicatrix for the line at infinity of T regarded as a line configuration.*

This outline may suffice to bring out the most prominent features of the analogy between point and line transformations.

§ 21. *Extensions and Generalizations.*

We have expressly assumed in the preceding investigations that our transformation is *of multiplicity one-to-one.* I shall reserve for a subsequent paper the detailed study of *transformations of higher multiplicity.* Nevertheless I think this is the proper place to point out that most of the results reached apply without modifications to the general case.

Let

$$U(\bar{x}, \bar{y}, x, y)=0, \quad V(\bar{x}, \bar{y}, x, y)=0, \tag{32}$$

be the equations of the transformation in implicit form, and let $P(x, y)$ be any point in the plane. If, regarding x and y as constants, the solution of equations (32) give a *discrete number of values for* $\bar{x}$ *and* $\bar{y}$ *all distinct* we shall say that the point P is *regular* with respect to the transformation T. If the point is not a regular point it is either a *singular* point, that is, it has an *infinity of corresponding points*, or it is a *point of coincidence*, that is, at least two of its images are coincident.

All our investigations conserving the indicatrix apply evidently to any continuous point to point transformation at any regular point. If the point P has n distinct images $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_N$, *there will be* n *indicatrices passing through* P, *one for each of its images.* Once the couple of points (x, y) and $(\bar{x}, \bar{y})$ determined, the equation of the indicatrix becomes

$$\begin{vmatrix} \frac{\partial U}{\partial x}(X-x)+\frac{\partial U}{\partial y}(Y-y), & \frac{\partial V}{\partial x}(X-x)+\frac{\partial V}{\partial y}(Y-y), \\ \frac{\partial U}{\partial \bar{x}}(X-\bar{x})+\frac{\partial U}{\partial \bar{y}}(Y-\bar{y}), & \frac{\partial V}{\partial \bar{x}}(X-\bar{x})+\frac{\partial V}{\partial \bar{y}}(Y-\bar{y}), \end{vmatrix}=0. \tag{33}$$

Assume now that the point P *is a point of coincidence.* For example, that T is a two-to-one transformation and the two images of P are coincident in $\bar{P}$. It is clear then that at P the image of any curve C passing through P will have a double point. To a ray t through P will correspond two rays through $\bar{P}$, but to a ray $\bar{t}$, but one ray t will correspond. The indicatrix is therefore cut by any ray through P in one point only. The indicatrix is therefore a cubic having a double point in $\bar{P}$ and passing through P. On the other hand, this cubic is but a *limiting* position of a configuration consisting of two conics, it must therefore degenerate and it consists of *three straight lines* α, β,

γ, of which α and β meet in $\overline{P}$. The point P is evidently on the Jacobian of T. At any other point of the plane the behavior of the indicatrix is regular, and *any determination of T can be regarded as a one-to-one transformation.* All the considerations developed in the general theory apply therefore with this restriction.

In the most general case of a continuous p-to-q transformation the behavior of the indicatrix at a point will be analogous to that of the example considered. As long as we avoid the singularities described above, all the results of the general theory hold.

* * *

As to the generalization of the theory, it can be pursued in two directions. First, we may extend the conception of the indicatrix to any *surface of genus zero;* second, to *space line-to-line transformations.* I reserve these considerations for a subsequent communication.

BLOOMINGTON, IND., *February*, 1917.

Properties of a Certain Projectively Defined Two-Parameter Family of Curves on a General Surface.

By Pauline Sperry.

1. *Analytic Foundation of the Differential Geometry of Non-Ruled Surfaces.*

In his first memoir on the "Projective Differential Geometry of Curved Surfaces,"* Mr. Wilczynski has shown that the projective theory of non-ruled analytic surfaces may be based on the consideration of a system of completely integrable partial differential equations of the second order which may be reduced to the so-called canonical form

$$y_{uu}+2by_v+fy=0, \quad y_{vv}+2a'y_u+gy=0, \tag{1}$$

where the subscripts denote partial differentiation, and where the coefficients are analytic functions of u and v satisfying the integrability conditions

$$\left.\begin{aligned} &a'_{uu}+g_u+2ba'_v+4a'b_v=0, \quad b_{vv}+f_v+2a'b_u+4ba'_u=0,\\ &g_{uu}-f_{vv}-4fa'_u-2a'f_u+4gb_v+2bg_v=0. \end{aligned}\right\} \tag{2}$$

Such a system of differential equations possesses exactly four linearly independent analytic solutions

$$y^{(k)}=f^{(k)}(u,v) \qquad (k=1,2,3,4). \tag{3}$$

If we now interpret $y^{(1)}, \ldots, y^{(4)}$, as the homogeneous coordinates of a point P_y, and let the independent variables range over all their values, P_y will generate a surface S_y, an integral surface of (1), which will be a ruled surface if, and only if, $a'=0$ or $b=0$ (a case which we shall exclude in this paper), and upon which the reference curves, $u=$constant and $v=$constant, are the

* There are five of these memoirs which appeared in the *Transactions of the American Mathematical Society* from 1907–1909. These will be referred to in the following pages as "First Memoir," etc.

asymptotic lines. Since the most general system of linearly independent solutions of (1) is of the form

$$\eta_i = \sum_{k=1}^{4} c_{ik} y^{(k)} \qquad (i=1, 2, 3, 4), \tag{4}$$

where $|c_{ik}| \neq 0$, the most general integrating surface of (1) is a projective transformation of any particular one.*

The canonical form is not uniquely determined. The most general transformation leaving it invariant will preserve the asymptotic curves as lines of reference and will be of the form

$$\bar{y} = C\sqrt{\alpha_u \beta_v}\, y, \quad \bar{u} = \alpha(u), \quad \bar{v} = \beta(v), \tag{5}$$

where α and β are arbitrary functions of u alone and of v alone respectively, and where C is an arbitrary constant.†

The functions

$$y = y, \quad y_u = z, \quad y_v = \rho, \quad y_{uv} = \sigma \tag{6}$$

are semicovariants. If the four independent solutions of (1) are substituted in z we get four functions $z^{(1)}, \ldots, z^{(4)}$, which may be taken as the homogeneous coordinates of a point P_z. So also for ρ and σ. The points P_y, P_z, P_ρ, P_σ are in general the vertices of a non-degenerate semicovariant tetrahedron T.‡ In just the same way every expression of the form

$$x = x_1 y + x_2 z + x_3 \rho + x_4 \sigma \tag{7}$$

determines a point P_x whose coordinates referred to T, by means of a suitable choice of the unit point, may be taken as (x_1, x_2, x_3, x_4).

2. *The Differential Equation of Certain Two-Parameter Families of Curves on a General Surface.*

The theory of two-parameter families of curves on a general surface has received but little attention except in so far as such a general theory may be implied by the theory of geodesics. We shall discuss in this paper a class of curves which will include the geodesics as a special case.

Let us associate with every point P_y of the surface one of the lines l_y which passes through that point, but does not lie in the tangent plane of the point. All these lines form a congruence L. Let us consider a curve on the surface which has the property that each of its osculating planes passes through the corresponding line of the congruence. All such curves will clearly

* First Memoir, p. 237. † First Memoir, pp. 90–95. ‡ Second Memoir, pp. 79–80.

form a two-parameter family, and it is easy to show that they will be the integral curves of an equation of the form

$$u''v'-u'v''+2(bu'^3-a'v'^3)+2(p_1u'^2v'+p_2u'v'^2)=0, \tag{8}$$

where $u'=du/dt$, $u''=d^2u/dt^2$, etc., and where p_1 and p_2 are functions of u and v which depend upon the choice of the congruence L.

The coordinates of P_y referred to the local tetrahedron of reference T are $(1, 0, 0, 0)$ so that the equation of any line l_y through P_y referred to T may be written in the form

$$A_2x_2+A_3x_3+A_4x_4=0, \quad B_2x_2+B_3x_3+B_4x_4=0, \tag{9}$$

where the coefficients are in general functions of u and v. If we denote by Π_1 and Π_2 the left-hand members of (9),

$$\lambda\Pi_1+\mu\Pi_2=0 \tag{10}$$

is the equation of any plane through l_y. If, in particular, this plane is the osculating plane of a surface curve through P_y, the coordinates of y' and y'' must also satisfy (10). By means of (6),

$$\left.\begin{aligned} y' &=dy/dt=u'z+v'\rho, \\ y''&=d^2y/dt^2=-(u'^2f+v'^2g)y+(u''-2a'v'^2)z+(v''-2bu'^2)\rho+2u'v'\sigma. \end{aligned}\right\} \tag{11}$$

The coordinates of y' and y'' referred to T are therefore

$$(0, u', v', 0), \quad (-u'^2f-v'^2g, \; u''-2a'v'^2, \; v''-2bu'^2, \; 2u'v').$$

The substitution of these coordinates in (10) gives upon the elimination of the ratio $\lambda:\mu$ equation (8), where

$$p_1=\frac{A_2B_4-A_4B_2}{A_3B_2-A_2B_3}, \quad p_2=\frac{A_3B_4-A_4B_3}{A_3B_2-A_2B_3}.$$

It is now evident that the line l_y must not lie in the tangent plane, as the denominator $A_3B_2-A_2B_3$ would vanish in that case and only in that case. If v be taken as the parameter of the curve, equation (8) will become

$$u''+2bu'^3+2p_1u'^2+2p_2u'-2a'=0. \tag{12}$$

It is also true that the one-parameter families of planes osculating the integral curves of an equation of the form (8) at the point P_y form a pencil. The equation of such a plane is

$$v'x_2-u'x_3+(p_1u'+p_2v')x_4=0. \tag{13}$$

If (u_1', v_1'), (u_2', v_2'), (u_3', v_3') are three pairs of values of u', v' corresponding to three distinct curves of this sort which pass through the same point, it is

evident that the corresponding equations (13) are not linearly independent for the determinant of the coefficients

$$\begin{vmatrix} v_1' & -u_1' & p_1u_1'+p_2v_1' \\ v_2' & -u_2' & p_1u_2'+p_2v_2' \\ v_3' & -u_3' & p_1u_3'+p_2v_3' \end{vmatrix}$$

vanishes identically. We have proved the following theorem:

If a two-parameter family of curves on a non-ruled surface has the property that the osculating planes of all of the curves of the family which pass through a given surface point P_y have in common a line through P_y, then the second order differential equation of the two-parameter family of curves has the form (8) *and conversely.*

These curves we have called *union curves* because of the characteristic property of united position of line and plane. It is evident that neither of the one-parameter families of asymptotic curves u=constant and v=constant can be union curves on a non-ruled surface since that would necessitate the condition $a'=0$, or $b=0$.

Among the congruences associated with the surface in the way described above are two of particular interest, the congruence of surface normals and the congruence of directrices of the second kind. One may define a geodesic as the curve whose osculating plane at a point contains the surface normal for that point. In case that for a particular one of the projectively equivalent integral surfaces of (1), the congruence L happens to be the congruence of surface normals, the corresponding union curves on that surface will be geodesics. Since perpendicularity is not a projective property, they would not in general be geodesics on the other integral surfaces of (1). The other congruence, the congruence of directrices of the second kind, is determined as follows: If we take five consecutive tangents to the curve v=constant, they determine in general a linear complex which approaches a limiting position as the tangents approach coincidence.* This complex is called the *osculating linear complex* for the asymptotic curve of the first kind. Similarly for the asymptotic curve of the second kind u=constant.† These complexes have as their intersection a congruence, one of whose directrices lies in the tangent plane of the point P_y, while the other, known as *directrix of the second kind,*

* E. J. Wilczynski, "Projective Differential Geometry," p. 162. In the following pages this book will be referred to as Proj. Diff. Geom.

† Second Memoir, pp. 90–95.

passes through the point itself, but does not lie in the tangent plane. The equations of the latter referred to the local tetrahedron T are

$$2bx_2+b_vx_4=0, \quad 2a'x_3+a'_ux_4=0. \tag{14}$$

If the congruence L is the directrix congruence of the second kind,

$$p_1=-\alpha, \quad p_2=\beta, \tag{15}$$

where

$$\alpha=a'_u/2a', \quad \beta=b_v/2b. \tag{16}$$

Equation (12) is a non-linear, non-homogeneous equation of the second order whose coefficients are functions of u and v. Since the complete solution involves two arbitrary constants, it represents a two-parameter family of curves, any one of which is uniquely determined when the surface point through which it must pass and the tangent at the point are given. The differential equation (12) can be integrated only in a few cases. The integration of an equation of this type in certain instances has been discussed by Darboux,* Liouville,† and Guldberg.‡ If L is the directrix congruence we can find a class of surfaces for which we can always integrate (12). Mr. Wilczynski has considered a class of surfaces for which the invariants $I=B_u/4BA^{\frac{1}{2}}$, and $J=A_v/4AB^{\frac{1}{2}}$, where $A=a'b^2$ and $B=a'^2b$, vanish identically, and he has shown that these surfaces are self-projective.§ For such surfaces we may assume without loss of generality that $a'=b=1$. Then equation (12) becomes $p'+2p^3-2=0$, where $p=u'$, and its first integral is

$$\log c\sqrt[6]{(p-1)^2/(p^2+p+1)}-(\sqrt[3]{3}/3)\ \text{arc tan}\ (2p+1)/\sqrt{3}=-2v.$$

3. *The Definition of Torsal Curves and Their Relation to Union Curves.*

There are two well-known fundamental properties of congruences. First, the lines of a congruence are the common tangents of two surfaces, or more precisely, they are the double tangents of a surface with two sheets, the focal surface. Second, if the two sheets of the focal surface do not coincide point for point, the lines of the congruence may be assembled into two one-parameter families of developables. We shall determine the curves on S_y such that the ruled surface composed of the lines of L, corresponding to the points of these curves shall be developables. These curves we shall call *torsal curves.* There

---

* Darboux, *Leçons*, Vol. III, Chap. 5.

† R. Liouville, "Sur une classe d'équations différentielles," *Comptes Rendus*, Vol. CV (1887), p. 1062.

‡ A. Guldberg, "On Geodesic Lines on Special Surfaces," *Nyt Tydsskrift. Math.*, Vol. VI (1895). (See *Jahrbuch*, "Ueber die Fortschritte der Mathematik.")

§ E. J. Wilczynski, "On a Certain Class of Self-Projective Surfaces," *Transactions of the American Mathematical Society*, Vol. XLV (1913), pp. 421–443.

will be two of these curves passing through each non-special point of the surface, one from each family. We shall assume, therefore, that the two sheets of the focal surface are distinct, that is, that it will be impossible to find two functions $w_1(u, v)$ and $w_2(u, v)$ such that

$$w_1(u, v)\xi^{(k)}+w_2(u, v)\eta^{(k)}=0 \qquad (k=1, 2, 3, 4),$$

where $\xi^{(k)}$ and $\eta^{(k)}$ are the coordinates of points on S_ξ and S_η, the two sheets of the focal surface.*

The equations of l_y, as given in (9), are satisfied by the coordinates $(0, -p_2, p_1, 1)$ so that the point P_t, where

$$t=-p_2z+p_1\rho+\sigma \tag{17}$$

is a point of l_y. If we allow u and v to take on the infinitesimal increments du and dv, the point P_y moves to the point $P_{y+y_udu+y_vd_v}$ and the point P_t to $P_{t+t_udu+t_vd_v}$ where

$$t_u=Py+Qz+R\rho+S\sigma, \quad t_v=P'y+Q'z+R'\rho+S'\sigma, \tag{18}$$

and

$$\left.\begin{aligned} P&=fp_2+2bg-f_v, & P'&=-gp_1+2a'f-g_u,\\ Q&=4a'b-p_{2u}, & Q'&=-2a'p_1-g-2a'_u-p_{2v},\\ R&=2bp_2-f-2b_v+p_{1u}, & R'&=\quad 4a'b+p_{1v},\\ S&=p_1, & S'&=-p_2.\end{aligned}\right\} \tag{19}$$

An arbitrary point P_x on the line $l_{y+y_udu+y_vd_v}$ will be represented by

$$\lambda(y+y_udu+y_vdv)+\mu(t+t_udu+t_vdv),$$

and its coordinates referred to T will be

$$\left.\begin{aligned} x_1&=\lambda+\mu(Pdu+P'dv), & x_2&=\lambda du+\mu(-p_2+Qdu+Q'dv),\\ x_3&=\lambda dv+\mu(p_1+Rdu+R'dv), & x_4&=\mu(1+Sdu+S'dv).\end{aligned}\right\} \tag{20}$$

In order that P_x may also be a point of l_y, its coordinates must satisfy (9), that is,

$$\left.\begin{aligned} \lambda[A_2du+A_3dv]+\mu[(A_2Q+A_3R+A_4S)du+(A_2Q'+A_3R'+A_4S')dv]=0,\\ \lambda[B_2du+B_3dv]+\mu[(B_2Q+B_3R+B_4S)du+(B_2Q'+B_3R'+B_4S')dv]=0,\end{aligned}\right\} \tag{21}$$

whence

$$\begin{vmatrix} A_2du+A_3dv & (A_2Q+A_3R+A_4S)du+(A_2Q'+A_3R'+A_4S')dv \\ B_2du+B_3dv & (B_2Q+B_3R+B_4S)du+(B_2Q'+B_3R'+B_4S')dv \end{vmatrix}=0. \tag{22}$$

* E. J. Wilczynski, "Sur la theorie general des congruences," Bruxelles, 1911.

If we let

$$L=2bp_2-f-p_1^2-2b_v+p_{1u},\quad 2M=p_{1v}+p_{2u},\quad N=2a'p_1+g+p_2^2+2a'_u+p_{2v},\tag{23}$$

then (22) becomes

$$Ldu^2+2Mdudv+Ndv^2=0;\tag{24}$$

this is the quadratic equation of the tangents to the torsal curves. If these curves form a conjugate system, there exists a function θ, such that $p_1=-\theta_u$ and $p_2=\theta_v$. When L is the directrix congruence of the second kind, the torsal curves are the directrix curves.* If L is the congruence of normals, they are the lines of curvature.

The question naturally arises whether some or all of the union curves might be plane. We shall show that this can happen if, and only if, they are at the same time torsal curves.

Every curve in three-dimensional space is characterized by a linear differential equation of the fourth order of the form,†

$$q_0y^{(\mathrm{IV})}+4q_1y'''+6q_2y''+4q_3y'+q_4y=0,\tag{25}$$

where $y^{(i)}=d^iy/dt^i$ $(i=1, 2, 3, 4)$, and $q_0, \ldots, q_4$ are in general functions of t. We may regard the solutions of (25) as the homogeneous coordinates of a point in space. As t varies this point describes a curve. Since

$$\eta_i=\sum_{k=1}^{4}c_{ik}y_k \qquad (i=1, 2, 3, 4)$$

is the most general solution of (25), we may say the differential equation defines a set of projectively equivalent curves in three-dimensional space. These curves will be plane if, and only if, $q_0=0$. We shall indicate the method for computing the fourth order differential equation which characterizes the union curves and actually calculate the value of q_0. From equations (1) and (6), and those obtained from them by partial differentiation, and from the value of u'' given by (12), we find the following formulae in which v is chosen as independent variable:

$$\left.\begin{aligned} y' &=u'z+\rho, & y'' &=a_1y+a_2z+a_3\rho+a_4\sigma,\\ y''' &=b_1y+b_2z+b_3\rho+b_4\sigma, & y^{(\mathrm{IV})} &=c_1y+c_2z+c_3\rho+c_4\sigma, \end{aligned}\right\}\tag{26}$$

---

* Second Memoir, p. 116.

† Proj. Diff. Geom., Chap. 2.

where

$$\left.\begin{aligned}
&a_1=-fu'^2-g, \quad a_2=-2bu'^3-2p_1u'^2-2p_2u', \quad a_3=-2bu'^2, \quad a_4=2u',\\
&b_1=6bfu'^4+(6fp_1-f_u)u'^3+3(2bg+2fp_2-f_v)u'^2-3g_uu'-g_v,\\
&b_2=-fu'^3+12a'bu'^2,\\
&b_3=12b^2u'^4+(12bp_1-2b_u)u'^3+3(4bp_2-f-2b_v)u'^2-g,\\
&b_4=-8bu'^3-6p_1u'^2-6p_2u'+4a',\\
&c_1=[(2bg-f_v)b_4-fb_2+b_{1u}]u'-gb_3+(2a'f-g_u)b_4+b_{1v},\\
&c_2=(b_1+4a'bb_4+b_{2u})u'-2a'b_3-(g+2a'_u)b_4+b_{2v},\\
&c_3=[-2bb_2+b_{3u}-(f+2b_v)b_4]u'+b_1+4a'bb_4+b_{3v},\\
&c_4=(b_3+b_{4u})u'+b_2+b_{4v},
\end{aligned}\right\} \tag{27}$$

where $b_{1u},\ldots,b_{4u},b_{1v},\ldots,b_{4v}$ are found from the fifth to eighth equations of (27) by partial differentiation, and u'' is given by (12). The desired equation will be obtained by eliminating z, ρ, and σ between equations (26) which gives

$$\begin{vmatrix} y' & u' & 1 & 0 \\ y''-a_1y & a_2 & a_3 & a_4 \\ y'''-b_1y & b_2 & b_3 & b_4 \\ y^{(\mathrm{IV})}-c_1y & c_2 & c_3 & c_4 \end{vmatrix}=0. \tag{28}$$

Thus the coefficient of $y^{(\mathrm{IV})}$ is

$$q_0=-\begin{vmatrix} u' & 1 & 0 \\ a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix},$$

or, substituting from (27),

$$q_0=4u'^2(Lu'^2+2Mu'+N). \tag{29}$$

Since u' can not vanish, $q_0=0$ is equivalent to (24). Hence we have proved the following theorem:

A necessary and sufficient condition that the union curves be plane is that they shall also be torsal curves.

This necessitates two relations between p_1, p_2, a', b and their derivatives which are obtained by the substitution in (12) of each of the pairs of values of u' and u'' found from (23) and the equation obtained from it by differentiation. If we let

$$R^2=M^2-LN, \tag{30}$$

these relations are

$$\left.\begin{aligned} 2a'L^3+8bM^3+2M^2(L_u-2p_1L)+L^2(2p_1N+2p_2M+M_v+\tfrac{1}{2}N_u) \\ -LM(2M_u+L_v)-LN(6bM-\tfrac{1}{2}L_u)=0, \\ R(4bM^2+2bLMN-4p_1LM+2p_2L^2-LM_u+2ML_u-LL_v) \\ +L^2R_v-LMR_u=0. \end{aligned}\right\} \quad (31)$$

The second of these is satisfied identically when $R=0$, that is, when the two families of torsal curves coincide. For this case (31) reduces to the single condition

$$2a'L^2+2bM^3-2p_1LM^2+2p_2L^2M-LMM_u+L_uM^2+L^2M_v-LL_vM=0, \quad (32)$$

for particular values of p_1 and p_2, one must always show that these conditions are not inconsistent with the integrability conditions (2). This caution applies also to all similar situations which occur in the following pages.

Since by definition the osculating plane of a union curve at the point P_y contains the line l_y, and since the osculating plane of a plane curve is the same for all points of the curve, it is obvious geometrically that plane union curves are torsal curves, and that the developables are the planes of the curves themselves. If every curve of the two-parameter family of union curves is plane, it is evident that the torsal curves must be indeterminate, since there are only a single infinity of them. In that case we must have

$$L=M=N=0, \quad (33)$$

and conditions (31) would be satisfied identically.

In case all the union curves are plane, the lines of L for a restricted region R of the surface have a point in common as a simple geometric argument will show. Consider that part of a union curve C lying in R. It follows from the theory of differential equations that for R sufficiently small, there exist union curves joining two arbitrary points of C to a third point of R not on C. We obtain in this way an infinite number of triples of points for which the corresponding lines of L intersect in pairs. Since these lines are not all in the same plane, they must all pass through the same point, the edge of regression of all of the developables of the congruence.

The theorem proved above is of especial interest as it includes as a particular instance the well-known theorem that a geodesic is plane when, and only when, it is a line of curvature.

4. *The Principle of Duality Applied to Union Curves.*

The integral surface S_y of (1) may be regarded as the envelope of its tangent planes. It will then be represented by the partial differential equations.*

$$Y_{uu}-2bY_v+(2b_v+f)Y=0, \quad Y_{vv}-2a'Y_u+(2a'_u+g)Y=0, \tag{34}$$

where the solutions Y_i $(i=1, 2, 3, 4)$ are proportional to the homogeneous coordinates of the plane p_y tangent to the surface S_y at the point P_y. If we let

$$\bar{a}'=-a', \quad \bar{b}=-b, \quad \bar{f}=2b_v+f, \quad \bar{g}=2a'_u+g, \tag{35}$$

we may replace (34) by

$$Y_{uu}+2\bar{b}Y_v+\bar{f}Y=0, \quad Y_{vv}+2\bar{a}'Y_u+\bar{g}Y=0, \tag{36}$$

which is called the system adjoined to (1). It is evident from (35) that each is the adjoint of the other. If the solutions Y_i $(i=1, 2, 3, 4)$ are regarded as the coordinates of a plane, systems (1) and (36) have the same integral surfaces. But if they are regarded as the coordinates of a point, every integral surface S_Y of (36) would be a dualistic transform of every integral surface S_y of (1). Since $a'\neq 0$ and $b\neq 0$, S_y is not identically self-dual, that is, there exists no dualistic transformation carrying the point P_y over into the plane p_y tangent at that point. If Y_i be interpreted as the coordinates of a point, then the equation of the union curves on S_Y is

$$u''-2bu'^3+2\bar{p}_1u'^2+2\bar{p}_2u'+2a'=0. \tag{37}$$

In order to pass to the dualistic interpretation, let us regard S_y as the locus of its points, and S_Y as the envelope of its tangent planes. To the congruence L of lines l_y passing through the points of S_y, will correspond a congruence L_Y of lines l_Y in the tangent planes of S_Y. To a curve as point locus on S_y will correspond a developable circumscribed about the surface S_Y. Just as the union curves on S_y have the property that the osculating plane at P_y determined by three consecutive points of the curve contains the line l_y, so the point, which for symmetry we shall call the *osculating point for* p_y, determined by three consecutive planes of the developable, lies on the line l_Y. As the line l_y is the intersection of the osculating planes of all of the union curves passing through P_y, so also l_Y is the locus of all the osculating points of the developables containing p_y. To a plane union curve would correspond a cone with the osculating point as vertex. The theorems developed in the preceding pages are capable of dualistic interpretation, and could be developed independently by analysis dualistic to that employed above. The values of $\bar{p}_1$ and $\bar{p}_2$

* First Memoir, p. 257.

in (37) evidently depend upon the choice of L_Y. As instance of projectively related congruences of this character one may cite the directrix congruences of the first and second kind.

5. *The Ruled Surface of the Congruence L along a Union Curve.*

We shall determine the differential equations which characterize the ruled surface generated by the line l_y as P_y moves along one of the union curves. The coordinates of a point t of the line l_y given in (17), and of y, must satisfy differential equations of the form*

$$n_{11}y''+p_{11}y'+p_{12}t'+q_{11}y+q_{12}t=0, \quad n_{22}t''+p_{21}y'+p_{22}t'+q_{21}y+q_{22}t=0, \tag{38}$$

where $y'=dy/dv$, $y''=d^2y/dv^2$ and so on. By means of (1), (12), (17), (18) and (19) we find

$$\left.\begin{aligned} &y=y, \quad y'=u'z+\rho, \quad y''=-(u'^2f+g)y+(u''-2a')z-2bu'\rho-2u'\sigma,\\ &t=-p_2z+p_1\rho+\sigma,\\ &t'=(u'P+P')y+(u'Q+Q')z+(u'R+R')\rho+(u'S+S')\sigma,\\ &t''=T_1y+T_2z+T_3\rho+T_4\sigma, \end{aligned}\right\} \tag{39}$$

where $T_1, \ldots, T_4$ are somewhat lengthy expressions in a', b, their partial derivatives and the powers of u' up to the third. By eliminating z, ρ and σ by means of the second, third, fourth and fifth of equations (39), and then from the second, fourth, fifth and sixth, there result two equations of type (38) where $n_{11}=n_{22}=-(Lu'^2+2Mu'+N)$. Since the ruled surface is a developable if, and only if, $n_{11}=n_{22}=0$, the results of § 3 are again emphasized.

The four pairs of independent solutions of (38), y_i, t_i, $(i=1, 2, 3, 4)$ may be taken as the homogeneous coordinates of two points P_y and P_t. As the line l_y generates the ruled surface, these points describe curves C_y and C_t. The curve C_y is asymptotic for that surface provided that $p_{12}=0$.† The calculation of the coefficients gives

$$p_{12}=4u'(p_1u'+p_2). \tag{40}$$

Since $u'\neq 0$, $p_{12}=0$ when, and only when, $u'=-p_2/p_1$. This value of u' must satisfy (12) which imposes the following restriction

$$p_1p_2(p_{2u}+p_{1v})-p_2^2(p_{1u}+2bp_2)-p_1^2(p_{2v}+2a'p_1)=0. \tag{41}$$

That this condition may be satisfied is apparent. Let L be the congruence of directrices of the second kind, and let a' and b be functions of v alone, and of

* Proj. Diff. Geom., p. 126.

† Proj. Diff. Geom., p. 142.

u alone, respectively. Then $p_1=p_2=0$. Whenever p_1 and p_2 are such that (41) is satisfied, there exists a one-parameter family of union curves having the property that they are at the same time asymptotic curves of the ruled surface generated by the lines of L along those curves.

6. *Some Remarks on a Problem in the Calculus of Variations.*

One of the most interesting properties of geodesics is that they appear in the problem of determining the lines of shortest length on a surface. Because so much of the theory of geodesics has turned out to be capable of generalization to the theory of union curves, the question naturally arises whether there exists a function $F(u, v, u')$ such that the integral $\int F(u, v, u')dv$ assumes a minimum value along a union curve, and such that the integral is invariant under the transformation $\bar{u}=\alpha(u)$, $\bar{v}=\beta(v)$. It is possible to find an infinity of functions satisfying the first of these conditions. The latter condition is essential in order to obtain an integral which may have an intrinsic projective significance. Investigation of this question has yielded thus far only negative results.

*Interpolation Properties of Orthogonal Sets of Solutions of Differential Equations.**

BY O. D. KELLOGG.

As the present paper is of the nature of a continuation of the one published in the last number of this Journal, an extended introduction may be dispensed with. Suffice it to say that we shall here be concerned with the problem of extending to the orthogonal function sets arising from ordinary differential equations of second order, the properties there derived for sets arising from integral equations.

1. *The Differential Equation and Boundary Conditions.*

We shall be concerned with differential equations of the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+(q(x)+\lambda r(x))y=0, \qquad p>0 \text{ for } 0<x<1, \quad r>0 \text{ for } 0\leqq x\leqq 1, \tag{1}$$

with the general homogeneous self-adjoint boundary conditions

$$ay(1)+by'(1)=cy(0)+dy'(0), \quad a'y(1)+b'y'(1)=c'y(0)+d'y'(0). \tag{2}$$

If $p(x)$ vanishes at one of the end points of the interval $(0, 1)$, but in such a way that the differential equation has a solution which remains finite and different from zero, with a finite derivative in the neighborhood of this point, one of the boundary conditions (2) is to be replaced by the condition: $y(x)$ remains finite and different from 0 in the neighborhood of this point. If $p(x)$ vanishes at both end points, both equations (2) are to be replaced by this demand for both end points. Examples of these cases are: (a) the functions $J_0(a_0x)\sqrt{x}$, $J_0(a_1x)\sqrt{x}$, ..., where a_0, a_1, ... are the successive roots of the Bessel function of zero order $J_0(x)$; (b) the Legendre polynomials on the interval $(-1, 1)$.

* Presented to the American Mathematical Society, December 1, 1917.

The existence of solutions of the problem (1) and (2), and their oscillation properties have been extensively studied,* and we shall make use of some of the more common ones. Our interest will center rather in the interpolation properties, and some of the consequences of the property (D),† which it will be our object to establish.

By "harmonics" we shall understand solutions of the differential equation and boundary conditions. The corresponding "frequencies" are the values of λ for which these solutions are possible. There is an infinite sequence of frequencies, without finite limit point, and bounded below. We shall think of them written in ascending order of magnitude, a frequency being written twice if it corresponds to two harmonics. The harmonics are orthogonal; ‡ even two corresponding to the same frequency may be made orthogonal by a proper choice of the integration constants entering them. No harmonic vanishes simultaneously with its derivative. The zeros of any harmonic separate those of any harmonic with the same or lower frequency. In addition to these known facts, we assume explicitly (1) the continuity of the coefficients of the differential equation and of the derivatives involved, (2) that the boundary conditions are such that for any pair of harmonics

$$p(0)\begin{vmatrix}\phi_i(0), \phi_j(0)\\ \phi_i'(0), \phi_j'(0)\end{vmatrix} = p(1)\begin{vmatrix}\phi_i(1), \phi_i(1)\\ \phi_j'(1), \phi_j'(1)\end{vmatrix}, \tag{3}$$

and (3), that the i-th *harmonic has exactly i zeros in the interior of* $(0, 1)$ *for all i.* A harmonic is "even" or "odd" according as the number of interior zeros is even or odd.

2. *Types of Boundary Conditions.*

For what follows it will be convenient to separate the boundary conditions into types according to the behavior of the corresponding harmonics at the end points of $(0, 1)$. Consider first the case in which the determinant of coefficients in (2) $ab'-a'b$ vanishes. There will then be a relation of the form $a''\phi_i(0)+b''\phi_i'(0)=0$. From this it follows that the function

$$M(x)=p(x)\,[\phi_i'(x)\phi_j(x)-\phi_i(x)\phi_j'(x)]$$

* See Bôcher, "Leçons sur les méthods de Sturm," Borel monographs, Paris, 1917. Rich indications as to the literature are found in the footnotes.

† AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII, No. 1 (1916), particularly (1), (2), I, II, IV, V, VI and V'. Sturm (*Liouville's Journal*, Vol. I, p. 433), and Liouville (same Journal, Vol. I, p. 269) studied the interpolation problem, and the method of the latter has suggested that used here. But the only existing results appear to be confined to boundary conditions of Class I below, which does not include the periodic and other interesting cases.

‡ Orthogonal here in the sense that $\int_0^1 \phi_i(x)\phi_j(x)r(x)\,dx=0,\quad i\neq j.$

vanishes for $x=0$ for every i and j, and in consequence, by (3), also for $x=1$. We are thus led to the first class of boundary conditions, in which is also included the cases in which $p(x)$ vanishes at one or both end points.

Class I. $M(0)\equiv M(1)\equiv 0$. One boundary condition involves only one end point, and the other, only the other. Consider first the case in which $p(x)$ vanishes at an end point. Then under our assumption, all harmonics are different from 0 at that end point. Consider an end point at which $p(x)$ does not vanish, say $x=0$. Then, as $M(0)=0$, $\phi_i'(0)\phi_j(0)-\phi_i(0)\phi_j'(0)=0$, and since no harmonic vanishes simultaneously with its derivative, it follows that if any harmonic vanishes for $x=0$, all do. We conclude for Class I: *If any harmonic vanishes at an end point, all vanish there.*

Supposing the determinant $cd'-c'd=0$ leads to the same class. We may therefore assume that both this and $ab'-a'b$ are different from zero, and that $p(0)>0$ and $p(1)>0$. The conditions (2) may now be given the form

$$y\,(1)=ay(0)+by'(0), \tag{4_1}$$

$$y'(1)=cy(0)+dy'(0), \tag{4_2}$$

or

$$y\,(0)=\quad dy(1)-by'(1), \tag{5_1}$$

$$y'(0)=-cy(1)+ay'(1), \tag{5_2}$$

where, by (3) and (4), $\Delta=ad-bc=p(0)/p(1)>0$. These conditions characterize the second class of cases.

Class II. $M(0)\equiv M(1)\not\equiv 0$. The boundary conditions both involve both end points. $M(0)$ and $M(1)$ may both vanish, but not as a consequence of the boundary conditions. There will be three types to consider in this class.

(a) $b=0$. Then $ad>0$, and (4_1) shows that if $\phi_i(x)$ vanishes at either end point, it does at the other. The sign of d now becomes important.

(a_1) $b=0$, $a>0$, $d>0$. If $\phi_i(x)$ vanishes at the end points, (4_2) shows that $\phi_i'(0)$ and $\phi_i'(1)$ have the same signs, so that $\phi_i(x)$ must have an odd number of interior zeros. But (4_1) shows the converse. Hence, for this case: *All odd harmonics vanish, and all even harmonics are different from zero at both end points.* An example of this case is $y''+\lambda y=0$, $y(1)=y(0)$, $y'(1)=y'(0)$, with the harmonics 1, $\sin 2\pi x$, $\cos 2\pi x$, $\sin 4\pi x$, $\cos 4\pi x$, ...

(a_2) $b=0$, $a<0$, $d<0$. By similar reasoning, we conclude for this case: *All even harmonics vanish, and all odd harmonics are different from zero at both end points.* An example is $y''+\lambda y=0$, $y(1)=-y(0)$, $y'(1)=-y'(0)$, with the harmonics $\sin \pi x$, $\cos \pi x$, $\sin 3\pi x$, $\cos 3\pi x$, ...

(b) $b>0$. Since a harmonic will not vanish simultaneously with its derivative, (4_1) shows that no harmonic vanishes at both end points. If

$\phi_i(0)=0$, (4_1) shows that $\phi_i(1)$ and $\phi_i'(0)$ have the same signs, and the harmonic must be even. The same conclusion follows from (5_1) when $\phi_i(1)=0$. Hence in this case: *No odd harmonic vanishes at either end point, and no harmonic vanishes at both.* An example is $y''+\lambda^2 y=0$, $\pi y(1)=2y'(0)$, $2y'(1)=-\pi y(0)$. The harmonics are, for $i\geqq 2$, $\phi_i(x)=\sin\lambda_i(x-1)+\dfrac{2}{\pi}\cos\lambda_i x$, where λ_i is the i-th positive root of $4\pi\lambda=(\pi^2+4\lambda^2)\sin\lambda$; while $\lambda_0=\lambda_1=\pi/2$. For these frequencies we have the solutions $A\cos\dfrac{\pi}{2}x+B\sin\dfrac{\pi}{2}x$. It will be seen if A and B are so chosen as to give an odd harmonic, this harmonic will be different from zero at both end points. All later harmonics are different from zero at both ends.

(c) $b<0$. By similar reasoning we conclude: *No even harmonic vanishes at either end point, and no harmonic does at both.*

3. *Lemmas.*

We shall need the following:

1. *If two harmonics have the same frequency, the one with the greater number of interior zeros is different from zero at both end points.* For the zeros of the two separate each other.

2. Let $y(x)=c_0\phi_0(x)+c_1\phi_1(x)+\ldots+c_n\phi_n(x)$ be a linear combination of harmonics, in which at least one of the constants c_i before the last is not 0. *If, for $c_n=0$, $y(x)$ has p interior roots, then for some $c_n\neq 0$, it will have at least p interior roots. If for $c_n=0$ it has an infinite number of interior roots, then for some $c_n\neq 0$, it will have at least N interior roots, N being any positive integer.* The proof is easily supplied by considering the limit of $y(x)$ as $c_n\doteq 0$, the approach of $y(x)$ to its limit being uniform. We shall reserve the notation $y(x)$ for functions of the above form.

3. *Given a function $y(x)$, $c_n\neq 0$, c_n remaining fixed, we can choose $c_0, c_1, \ldots, c_{n-1}$ so small, not necessarily zero, that $y(x)$ has the same number of interior zeros as $\phi_n(x)$ (i. e., exactly n), provided $\phi_n(x)$ does not vanish at an end point at which an earlier harmonic is different from zero. And in any case, $y(x)$ has not more than one additional interior zero corresponding to a terminal zero of $\phi_n(x)$.* The proof depends upon the facts that $\phi_n(x)$ does not vanish simultaneously with its derivative, and that all the harmonics are finite, with finite derivatives.

4. *Establishment of the Property (D).*

The results and treatment differ for the different types of boundary conditions enumerated. But the general argument is this. With a solution $\psi(x)$ of the differential equation, corresponding to boundary conditions to be assigned later, and not vanishing in the interior of $(0, 1)$, we form the differential operators to be applied to $y(x)$:

$$Ny(x) = p(x)[y'(x)\psi(x) - y(x)\psi'(x)], \text{ and } Ly(x) = \frac{d}{dx}Ny(x). \tag{6}$$

It will first be shown that repeated application of the operator L reduces relatively the earlier coefficients of $y(x)$ to any desired degree, and thus, by the third lemma, leads to a function with no more than n interior roots. It will then be shown that the application of the operator L does not diminish the number of interior roots. The result, with the modifications noted, will be that no $y(x)$ has more than n interior roots, and accordingly, that the determinant of the equations $y(x_0) = 0$, $y(x_1) = 0, \ldots, y(x_n) = 0$ does not vanish for any set of arguments x_i, no two of which are equal. The property (D) will follow (see this Journal, Vol. XL, p. 146, footnote).

Note first, however, that $D_0(x_0) > 0$, since the first harmonic has no interior zero. Second, that in case $\lambda_1 = \lambda_0$, $D_1(x_0, x_1) > 0$. For the two harmonics, belonging to the same frequency, will have a non-vanishing Wronskian, from which fact the desired result may be inferred. We may therefore suppose in the following, $n \geqq 1$, or, in case $\lambda_1 = \lambda_0$, $n \geqq 2$.

The choice of $\psi(x)$ in the operators (6) may vary with the function $y(x)$ to which it is to be applied, and the corresponding parameter value λ^* for which $\psi(x)$ is a solution of the differential equation (1) will vary with it. We shall assume, however, that there are two constants, κ and ρ, such that always

$$\kappa \leqq \lambda^* \leqq \rho \text{ where } \rho < \frac{\lambda_0 + \lambda_1}{2}, \text{ or if } \lambda_1 = \lambda_0, \ \rho < \frac{\lambda_0 + \lambda_2}{2}. \tag{7}$$

The assumption will be considered below as use is made of it. Using the differential equation, it will be found that

$$Ly(x) = p(x)\psi(x)r(x)[c_0(\lambda^* - \lambda_0)\phi_0(x) + c_1(\lambda^* - \lambda_1)\phi_1(x) + \ldots + c_n(\lambda^* - \lambda_n)\phi_n(x)].$$

Dividing, on the assumption $c_n \neq 0$, by $p(x)\psi(x)r(x)(\lambda^* - \lambda_n)$ we arrive at a new function

$$y(x) = c_0\left(\frac{\lambda^* - \lambda_0}{\lambda^* - \lambda_n}\right)\phi_0(x) + c_1\left(\frac{\lambda^* - \lambda_1}{\lambda^* - \lambda_n}\right)\phi_1(x) + \ldots + c_{n-1}\left(\frac{\lambda^* - \lambda_{n-1}}{\lambda^* - \lambda_n}\right)\phi_{n-1}(x) + c_n\phi_n(x).$$

Under the assumption (7) the quantities $\left(\frac{\lambda^*-\lambda_i}{\lambda^*-\lambda_n}\right)$ are all less in absolute value than certain quantities independent of λ^* and less than 1, an exception arising only in the case of the last of them when $\lambda_n=\lambda_{n-1}$. Hence, by repeating the application of L and the division, we may reduce all the coefficients of $y(x)$ except the last (or, in case $\lambda_n=\lambda_{n-1}$, the last two) to be less than any assigned positive quantity. This completes our first step.

The next step is to study how this process affects the number of interior zeros in the various cases enumerated.

CLASS I. Choose $\psi(x)=\phi_0(x)$. Then $\lambda^*=\lambda_0$, and (7) is satisfied. If $y(x)$ has p interior zeros, $Ny(x)$ will be seen to vanish between each pair, and thus have at least $p-1$ interior zeros. But since in this case $M(x)$ vanishes at both end points for all pairs of harmonics, and as $y(x)$ is linear in the harmonics, it follows that $Ny(x)$ also vanishes at both end points, and hence $Ly(x)$ has at least p interior zeros. Thus no interior zeros are lost by the repeated application of L. Lemma 2 permits us to assume $c_n\neq 0$, and then Lemma 3 shows that no $y(x)$ has more than n interior zeros, since by § 2, no harmonics vanish at an end point under the present boundary conditions unless all do. And a case of double frequency does not arise.* *Hence, in problems of Class I, the property* (*D*) *holds.*

CLASS II, (a_1). $b=0$, $a>0$, and $y(1)=ay(0)$, for $y(x)$ is a linear homogeneous combination of harmonics, and hence satisfies the same boundary conditions. Hence $y(0)$ and $y(1)$, if not both zero, have the same signs. We again use $\psi(x)=\phi_0(x)$, and assume that the last harmonic in $y(x)$, $\phi_n(x)$ is even. If $y(x)$ has an odd number, $2p-1$, of interior zeros, it also has two at the end points, and $Ny(x)$ has therefore at least $2p$ interior zeros, and $Ly(x)$ at least $2p-1$. Thus, if the application of L to $y(x)$ changes this function from one with an odd number of interior zeros to a new one with an odd number, none are lost. If it changes $y(x)$ to a function with an even number of interior zeros, at least one is *gained*. If $y(x)$ has an even number, $2p$, of interior zeros, $Ny(x)$ has at least $2p-1$, but (3) shows that it then either has an additional interior zero, or else two at the end points, and in either case $Ly(x)$ has at least $2p-1$ interior zeros. If $Ly(x)$ has an even number of interior zeros, it must be at least $2p$, and if odd, at least $2p-1$. But ultimately, by Lemma 3, the repeated application of L leads to a function with an even number of interior zeros, since we have supposed this to be the case with

* For two harmonics of the same frequency, $M(x)=$const. This constant is not zero if the harmonics are independent. But it must be 0 in the problems of Class I.

$\phi_n(x)$. Hence no zeros have been lost, and we conclude: *The harmonics of Case II* (a_1) *have the property* (D) *for every even* n.

The conclusion is still justified in case $\lambda_{n-1}=\lambda_n$. Here we must study $c_{n-1}\phi_{n-1}(x)+c_n\phi_n(x)$. If this has an even number of interior zeros, their number can not exceed n, since they are separated by the zeros of $\phi_{n-1}(x)$, and $c_{n-1}\phi_{n-1}(x)+c_n\phi_n(x)$ will, by the boundary conditions, be different from 0 at the end points. The conclusion thus subsists in this case. If $c_{n-1}\phi_{n-1}(x)+c_n\phi_n(x)$ has an odd number of interior zeros, the repeated application of L to $y(x)$ may have reduced the number of interior zeros of $y(x)$ by one, but by no more, and $c_{n-1}\phi_{n-1}(x)+c_n\phi_n(x)$ has $n-1$ interior zeros, and none at the end points, since by § 2, $\phi_{n-1}(x)$ vanishes at both end points, and $\phi_n(x)$ at neither, and $c_n\neq 0$. The application of the lemma 3 then shows that $y(x)$ can not have more than n interior zeros, and the conclusion is established also in the case $\lambda_{n-1}=\lambda_n$.

Class II, (a_2). $b=0$, $a<0$, and $y(1)=ay(0)$, so that if $y(1)$ and $y(0)$ are different from zero, they have opposite signs. We now operate with L in which $\psi(x)$ is chosen subject to the condition that $y'(x)\psi(x)-y(x)\psi'(x)$ vanishes at both end points. By § 2, $\phi_0(x)$, being an even harmonic, vanishes at both end points, and as $\psi(x)$ is different from zero on the interior of $(0, 1)$, it follows that $\lambda^*\leqq\lambda_0$. Hence the assumption (7) is justified as far as the upper limit on λ^* is concerned. The existence of the lower limit will be generally established at the end of the paper. We start with a function $y(x)$ ending in an odd harmonic. The operator N does not reduce the number of interior zeros by more than one, while with the present choice of $\psi(x)$, $Ny(x)$ has zeros at the end points. So the operator L does not diminish the number of interior roots. As the odd harmonic $\phi_n(x)$, by § 2, is different from 0 at the end points, Lemma 3 gives the result: *The harmonics of Case II* (a_2) *have the property* (D) *for every odd* n.

Cases II, (b) and (c). Here, again, we choose $\psi(x)$ so as to make $Ny(x)$ vanish at the end points. It is not, in these cases, apparent that the upper limit for λ^* in (7) holds. However, since $\psi(x)$ does not vanish within $(0, 1)$, and as $\phi_2(x)$ vanishes twice, it is clear that $\lambda^*<\lambda_2$. As the frequencies have no finite limit point, there must be a positive integer n, and a number ρ, independent of λ^*, such that $\lambda^*\leqq\rho<\dfrac{\lambda_0+\lambda_n}{2}$. The least value of n for which these inequalities are possible,* we will call n'. The type of argument used in the

* This least value of n can often be easily determined in a particular problem, it being helpful to observe that the value of λ for which the differential equation has a solution vanishing at the end points of the interval (0, 1) but not in the interior, is an upper bound for λ^*. In the example in § 9 under Case II (b), (7) is fulfilled without alteration.

previous cases then permits us to conclude: *The harmonics of Case II (b) have the property (D) for every odd $n \geqq n'$, and, provided none of the harmonics after $\phi_{n'-1}(x)$ vanishes at either end point, for every $n \geqq n'$. Also, the harmonics of Case II (c) have the property (D) for every even $n \geqq n'$, and, provided none of the harmonics after $\phi_{n'-1}(x)$ vanishes at either end point, for every $n \geqq n'$.*

5. *Existence of a Lower Bound for λ^*.*

In the problems of Class II, from (a_2) inclusive, on, it was assumed that a lower bound for the values of λ^* existed. We proceed to show that such is the case. It will be recalled that $\psi(x)$ was to be chosen so that the ratios $\psi'(1)/\psi(1)$ and $\psi'(0)/\psi(0)$ coincided with the corresponding ratios for $y(x)$. Let us call these ratios η and ξ respectively. They are not independent, but are connected by the equation, obtained by dividing (4_2) by (4_1):

$$\eta=(c+d\xi)/(a+b\xi), \quad \text{where } ad-bc>0. \tag{8}$$

We shall now prove the theorem:

There exists a number Λ, independent of ξ and η, such that the solution of the differential equation (1) *which satisfies the boundary conditions*

$$u'(0)=\xi u(0), \quad u'(1)=\eta u(1), \tag{9}$$

subject to the relation (8), *and which does not vanish in the interior of* $(0,1)$, *belongs to a parameter value $\lambda^*>\Lambda$.*

We assume that $p(0)>0$ and $p(1)>0$. The theorem is not needed in the problems of Class I.

We first prove the lemma: *For fixed ξ and decreasing η, and for fixed η and increasing ξ, the value of the parameter λ corresponding to* (9) *increases.* Let $u_1(x)$ and $u_2(x)$ be two solutions of the differential equation satisfying the first equation (9). Denoting the corresponding parameter values by λ_1 and λ_2, we find the following identity:

$$p(1)\,[u_2(1)u_1'(1)-u_1(1)u_2'(1)]=(\lambda_2-\lambda_1)\int_0^1 u_2(x)\,u_1(x)\,r(x)\,dx.$$

We are, in this paragraph, concerned only with solutions of (1) which do not vanish in the interior of $(0, 1)$, so that it is legitimate to consider $u_1(x)$ and $u_2(x)$ positive for $0<x<1$, so that the integral is positive. Dividing by $u_2(1)u_1(1)$, we have

$$p(1)\,[\eta(\lambda_1)-\eta(\lambda_2)]=\frac{(\lambda_2-\lambda_1)}{u_2(1)\,u_1(1)}\int_0^1 u_2(x)\,u_1(x)\,r(x)\,dx, \tag{10}$$

which shows that decreasing η corresponds to increasing λ. The second part of the lemma is proved similarly.

We next find the relation between ξ and η when these are the values of the ratios $u'(0)/u(0)$ and $u'(1)/u(1)$ corresponding to any *fixed* λ. To do this we take a fundamental solution set for $x=0$, $u_1(x)$ and $u_2(x)$, for which $u_1(0)=1$, $u_1'(0)=0$, $u_2(0)=0$, $u_2'(0)=1$, in terms of which any solution $u(x)$ for the same value of λ may be expressed in the form $u(x)=c_1 u_1(x)+c_2 u_2(x)$. Then $\xi=c_2/c_1$, and $\eta=[c_1u_1'(1)+c_2u_2'(1)]/[c_1u_1(1)+c_2u_2(1)]$. Eliminating the constants, we find the relation

$$\eta=\frac{u_1'(1)+\xi u_2'(1)}{u_1(1)+\xi u_2(1)}, \tag{11}$$

for which we have the easily derived relation

$$u_1'(1)u_2(1)-u_1(1)u_2'(1)=p(0)/p(1)>0.$$

This shows that the hyperbola which is the Cartesian representation of the equation (11) is an always rising one. For various values of λ, this hyperbola varies in position and size, but if for a finite value Λ of λ, its center lies above and to the left of the hyperbola (8), also an always rising hyperbola, the lemma shows that the solution of the differential equation and the boundary conditions (9) subject to (8) will always correspond to a value of $\lambda>\Lambda$, and the theorem will be proved.

That for some finite λ the center of the hyperbola (11) lies above and to the left of the upper left-hand branch of the hyperbola (8), will be evident from the fact which we shall next prove: given any finite positive quantities h and k, a finite λ always exists for which the coordinates of the center of the hyperbola (11) satisfy the inequalities

$$u_1(1)/u_2(1)>h, \quad u_2'(1)/u_2(1)>k. \tag{12}$$

The identity $\dfrac{u_2(1)}{u_1(1)}=\displaystyle\int_0^1\frac{p(0)}{p(x)u_1^2(x)}\,dx$, the derivation of which gives no trouble if the initial values of the fundamental solution set are kept in mind, shows that the ratio $u_1(1)/u_2(1)$ can be made as great as we wish if $u_1(x)$ can be made as great as we please at all interior points. But this fact can be inferred from the identity

$$\frac{d}{dx}\left(\frac{u_1(x,\lambda)}{u_1(x,\lambda_1)}\right)=\frac{\lambda_1-\lambda}{p(x)u_1^2(x,\lambda_1)}\times\int_0^x u_1(x,\lambda)u_1(x,\lambda_1)r(x)\,dx.$$

This shows that $u_1(x,\lambda)/u_1(x,\lambda_1)$ increases with x if $\lambda<\lambda_1$. This ratio

approaches 1 as $x \doteq 0$. Hence always, $u_1(x,\lambda) > u_1(x,\lambda_1)$ for $0<x<1$, and using the law of the mean, $\frac{d}{dx}\frac{u_1(x,\lambda)}{u_1(x,\lambda_1)} > \frac{\lambda_1-\lambda}{\text{max. } p}\int_0^x r(x)\,dx > (\lambda_1-\lambda)\times$ a positive function of x independent of λ. Hence the ratio, and therefore $u_1(x,\lambda)$ can be made as large as we please, and the first inequality (12) is established.

To prove the second, we start by noticing that for sufficiently small λ, $u_2'(x)>0$ for $0<x<1$. This is seen from the equation $p(x)u_2'(x)=p(0)+\int_0^x(-\lambda r-q)u_2\,dx$, in which the integral is positive if $\lambda<-\text{max. } q/\text{min. } r$. Multiplying the differential equation for u_2 by $2p(x)u_2'(x)$ and integrating, we have $[pu_2'(x)]^2\Big|_0^1=-2\int_0^1 p(\lambda r+q)u_2u_2'\,dx=-\overline{p(\lambda r+q)}\,u_2^2(1)$, the bar denoting a mean value. Hence $u_2'(1)/u_2(1)>\frac{1}{p(1)}\sqrt{(-\lambda)\text{ min. } pr-\text{max. } pq}$, which can be made as large as we please by sufficiently diminishing λ. Thus the second inequality (12) is proved.

COLUMBIA, MO., *October* 20, 1917.

Directed Integration.

By H. B. Phillips.

In case of an integral along a curve,

$$\int P dx + Q dy = \text{Lim}\, \Sigma P \Delta x + Q \Delta y,$$

the increments Δx and Δy may be positive or negative according as x and y are increasing or decreasing. In case of a double integral, however,

$$\int\int f(x, y)\, dxdy = \text{Lim}\, \Sigma\Sigma f(x, y)\, \Delta x \Delta y,$$

the element $\Delta x \Delta y$ is usually considered positive, or at least, invariable in sign. This introduces difficulties similar to those which occur when we attempt to banish the minus sign from algebra or analytic geometry. For instance, in a change of variable, it is necessary to assume that the Jacobian has an invariable sign.

Physicists avoid these difficulties by introducing a cosine which is positive or negative as required. Mathematicians accomplish the same result by making a change of variable, thus obtaining an element of integration which need not change sign. I wish to show in this paper how the algebraic sign can be directly attached to the element of integration, multiple integrals being treated in this respect like curvilinear integrals. The equations for change of variable and those connecting line, surface, and volume integrals, present themselves much more naturally in this form. In this discussion I shall not enter into questions of existence and convergence. These matters are treated in practically the same way whether the integral is directed or not. I shall also consider only two and three dimensions, although the extension to higher spaces is immediate.

Directed Regions.—A surface is called one-sided if it is possible to pass from a point on one side of the surface to a point on the other without passing through the surface or across its border. If this is not possible, the surface is called two-sided. A simple one-sided surface can be formed by twisting a strip of paper through 180° and bringing its ends together.

If a surface is two-sided, one side can be considered positive, the other negative. We shall assign a direction or sense to a region on a two-sided surface by assigning a direction around its border. In a right-hand system the positive direction is usually chosen such that an observer on the positive side of the surface finds the region on his left when he moves in the positive direction along the border. It should be noted that this direction does not belong to the border, but to the region of which it is the border. Thus a given direction around a great circle of a sphere is positive for the hemisphere on one side, and negative for that on the other.

Surface Integrals.—Let x and y be one-valued and continuous functions defined at each point of a region Γ on a plane or two-sided surface. Divide Γ into elementary regions, or cells, by two sets of curves

$$x=\text{constant}, \quad y=\text{constant}.$$

Any one of these cells whose boundary is a simple quadrilateral with two pairs of opposite sides belonging to the curves x, $x+\Delta x$ and y, $y+\Delta y$ will be called regular. Irregular cells may be bounded by less than four curves, or by four curves that are not of this simple type.

In the definition of the integral only the regular cells will be used. Hence it is assumed that the irregular cells can be enclosed in a region or set of regions whose total area approaches zero when Δx and Δy approach zero. This is certainly true of the curves ordinarily used in integration. It would not be true if the two systems of curves $x=$const. and $y=$const. were the same.

Choose a direction around one of the quadrilaterals. Then for that quadrilateral we define $\Delta x \Delta y$ as the product obtained by multiplying the increments of x and y which are found by passing around the quadrilateral in the chosen direction so as first to traverse a curve $y=$const., and then a curve $x=$const. The sign of $\Delta x \Delta y$ is fixed for a given quadrilateral and a given direction around it. Thus in the quadrilateral $ABCD$, if AB and DC are portions of curves $y=$const., and AD and BC portions of curves $x=$const., we may take Δx from A to B and Δy from B to C, or we may take Δx from C to D and Δy from D to A. In the second case the signs of Δx and Δy are both changed, and so $\Delta x \Delta y$ is not changed. Similarly, the product $\Delta y \Delta x$ is obtained by traversing the quadrilateral in the same direction, first traversing a curve $x=$const. and then a curve $y=$const. Inspection of a figure will make it clear that one of the increments in $\Delta y \Delta x$ differs in sign from the corresponding increment in $\Delta x \Delta y$. Hence

$$\Delta y \Delta x = -\Delta x \Delta y. \tag{1}$$

Let the same direction be taken around all the quadrilaterals into which Γ is divided. Then, if (x, y) is any point in the quadrilateral to which $\Delta x \Delta y$ belongs, we define the integral of $f(x, y)$,

$$\int\int f(x, y)\, dxdy,$$

in the chosen direction over Γ as the limit (if it exists) approached by the sum

$$\Sigma\Sigma f(x, y) \Delta x \Delta y$$

when Δx and Δy approach zero, the summation being for all the regular cells within Γ. Similarly,

$$\int\int f(x, y)\, dydx = \text{Lim}\ \Sigma\Sigma f(x, y) \Delta y \Delta x.$$

Therefore, by equation (1),

$$\int\int f(x, y)\, dydx = -\int\int f(x, y)\, dxdy. \tag{2}$$

It should be noted that the order in which the differentials are written does not indicate an order of integration. In fact, no order of integration is considered. The integrals are multiple, not iterated.

We have assumed that Δx is determined along the curves $y=$const., and Δy along the curves $x=$const. It is a very important fact that one of these increments could be determined along a third set of curves $w=$const. Thus, if we resolve Γ into cells by the curves $x=$const. and $w=$const., and in each quadrilateral determine Δy on $x=$const. as before, but Δx on $w=$const., the value of the integral will not be changed, provided the total area of the irregular cells formed by the new curves has a zero limit. For Γ can be resolved into strips between consecutive curves x, $x+\Delta x$. All the quadrilaterals in a strip have the same Δx. Also Δy is taken in both cases along the curves $x=$const. Hence, in the change assumed, the part of the summation belonging to this strip is affected only through the change in the distribution of the intervals Δy. This does not affect the limit.

Volume Integrals.—Triple integrals are defined in a similar way. Let x, y, z be one-valued and continuous functions defined at each point of a region Γ of space. Divide Γ into cells by means of three sets of surfaces $x=$const., $y=$const., and $z=$const. We shall call the cells regular which are bounded on opposite sides by three pairs of surfaces x and $x+\Delta x$, y and $y+\Delta y$, z and $z+\Delta z$. We assume that when Δx, Δy, and Δz approach zero, the total volume of the irregular cells approaches zero.

Let AB, BC, CD be consecutive edges of a cell, y and z being constant on AB, z and x on BC, x and y on CD. Let the outer surface of the cell be considered positive. The path BCD determines a direction (positive or negative)

about the face of the cell in which B, C, D lie. I call this the direction, or sense, of the cell $ABCD$.

For a given cell taken with an assigned direction, we define $\Delta x \Delta y \Delta z$ as the product obtained by multiplying the increments of x, y, z which are found by passing along consecutive edges of the cell in the assigned direction, first traversing an edge on which x alone varies, then one on which y alone varies, and finally one on which z alone varies. It is easy to verify that this fixes $\Delta x \Delta y \Delta z$ not only in magnitude, but also in sign when the cell is given and a direction assigned to it. Thus let the cell be a rectangular box $ABCDEFG$ formed by the parallel rectangles $ABCH$ and $FEDG$ (x varying on AB, y on BC, and z on CD). We may take Δx on AB, Δy on BC, and Δz on CB, or we may take Δx on EF, Δy on FG, and Δz on GH. In the second case two signs (those of Δx and Δz) are changed, and so $\Delta x \Delta y \Delta z$ is not changed.

Similarly, the product $\Delta y \Delta z \Delta x$ is obtained by traversing the edges of the cell in the same direction, first traversing an edge on which y alone varies, then one on which z alone varies, and finally, one on which x alone varies. The other products of Δx, Δy, Δz are defined in a similar way. It is easy to verify that each inversion of the order of Δx, Δy, Δz introduces a negative sign in the result. Thus

$$\Delta x \Delta y \Delta z = -\Delta x \Delta z \Delta y = \Delta z \Delta x \Delta y. \tag{3}$$

Let all the cells in a region Γ be taken in the same direction. If (x, y, z) is any point within the cell to which $\Delta x \Delta y \Delta z$ refers, the integral in the assigned direction over Γ,

$$\int\int\int f(x, y, z)\, dx dy dz,$$

is defined as the limit (if it exists) approached by

$$\Sigma\Sigma\Sigma f(x, y, z)\, \Delta x \Delta y \Delta z$$

when Δx, Δy, and Δz approach zero, the summation being for all the regular cells within Γ. Similarly,

$$\int\int\int f(x, y, z)\, dy dz dx = \mathrm{Lim}\ \Sigma\Sigma\Sigma f(x, y, z)\, \Delta y \Delta z \Delta x,$$

etc. Hence, from (3),

$$\int\int\int f(x, y, z)\, dx dy dz = -\int\int\int f(x, y, z)\, dx dz dy = \int\int\int f(x, y, z)\, dz dx dy. \tag{4}$$

As in case of double integration, one of the sets of surfaces $x=$const., $y=$const., $z=$const., can be replaced by a third set of surfaces, provided the irregular cells thus introduced have a total volume that approaches zero in the limit.

Expression of an Integral in Terms of Boundary Values.—A simple integral is expressed in terms of its limits by the formula

$$\int_a^b du = u \Big|_a^b = u(b) - u(a). \tag{5}$$

Similar formulas apply to multiple integrals.

Let u be a function of x and y. In the integral,

$$\int\int du dx,$$

let du be taken along $x=$const. We may take dx along $u=$const. or $y=$const. as we choose. The integral of du along a curve $x=$const. from one intersection with the border to another is given by equation (5). Hence, if we evaluate the integral by summing first with respect to u and then with respect to x, we get

$$\int\int du dx = \int u dx. \tag{6}$$

The double integral is taken over a region Γ, the simple integral over the boundary of Γ. Since du and dx occur on consecutive sides of a quadrilateral in the double integral, the direction of integration around the boundary must be such that if BC belongs to the boundary and $ABCD$ is a quadrilateral directed as in the double integral, then the integral along the boundary is in the direction BC.

Similarly, if u is a function of x, y, z,

$$\int\int\int du dx dy = \int\int u dx dy. \tag{7}$$

The triple integral is taken over a region Γ, the double integral over its boundary. The directions of integration are so related that if $ABCD$ is a cell of the triple integral with face BCD in the boundary, then BCD gives in that face the direction of the double integral.

Illustrations of these formulas are furnished by the theorems of Green, Stokes, and Gauss. Suppose, for example, P, Q, R are functions of x, y, z on a two-sided surface. Then

$$\int P dx = \int\int dP dx,$$

the two integrals being taken over a region of the surface and around its boundary, respectively. Since dP is determined on the curves $x=$const.,

$$dP = \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz,$$

and

$$\int P dx = \int\int \frac{\partial P}{\partial y} dy dx + \frac{\partial P}{\partial z} dz dx.$$

In these integrals dy and dz are taken along $x=\text{const.}$ In the combination $dydx$ we may take dx along $y=\text{const.}$ and in $dzdx$ along $z=\text{const.}$ Similarly,

$$\int Qdy=\int\int \frac{\partial Q}{\partial x}\,dxdy+\frac{\partial Q}{\partial z}\,dzdy,$$

$$\int Rdz=\int\int \frac{\partial R}{\partial x}\,dxdz+\frac{\partial R}{\partial y}\,dydz.$$

Adding these equations and changing the sign each time we invert the order of differentials, we get

$$\int Pdx+Qdy+Rdz=\int\int\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy+\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dzdx,$$

which is Stokes' Theorem. Since

$$dP=\frac{\partial P}{\partial x}\,dx+\frac{\partial P}{\partial y}\,dy+\frac{\partial P}{\partial z}\,dz,$$

the equation

$$\int\int dPdx=\int\int \frac{\partial P}{\partial y}\,dydx+\frac{\partial P}{\partial z}\,dzdx,$$

suggests that symbolically

$$\int\int \frac{\partial P}{\partial x}\,dxdx=0. \tag{8}$$

This is in line with our definition since one set of curves does not give rise to any regular quadrilaterals, and so the summation from which the integral might be defined is zero.

Change of Variable.—Let x, y be functions of u, v. Then

$$dx=\frac{\partial x}{\partial u}\,du+\frac{\partial x}{\partial v}\,dv$$

along any curve. Hence

$$\int\int fdxdy=\int\int f\frac{\partial x}{\partial u}\,dudy+\int\int f\frac{\partial x}{\partial v}\,dvdy.$$

In these integrals dy is determined along the curves $x=\text{const.}$ Since this is not significant we may determine dy in the first integral on the curves $v=\text{const.}$, and in the second integral on the curves $u=\text{const.}$ In the first case

$$dy=\frac{\partial y}{\partial v}\,dv,$$

and in the second

$$dy=\frac{\partial y}{\partial u}\,du,$$

whence

$$\int\int fdxdy=\int\int f\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\,dudv+\int\int f\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\,dvdu=\int\int f\left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right]dudv. \tag{9}$$

This result could be obtained by using the values

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv,$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv,$$

in $\int\int f dx dy$, expanding, and assuming that the integrals containing $du du$ and $dv dv$ are zero.

In a similar way we show

$$\int\int\int f dx dy dz = \int\int\int f \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} du dv dw. \tag{10}$$

Equations (9) and (10) are valid whether the Jacobian has an invariable sign or not. Irregular cells will, however, usually occur in the neighborhood of a point where the Jacobian is zero. It may therefore be necessary that it be possible to enclose such points in an area in case of a double integral and in a volume in case of a triple integral which has a limit zero.

While an integral is represented by a number, the summation process upon which the integral is based is as much geometric as arithmetic. Symbolically the integral is a function in which $dx dy$ is equivalent to $-dy dx$ and dx^2 to zero. This can be expressed by means of vectors. Yet dx, dy, dz are not vectors, and integration belongs no more to vector analysis than algebra does. They both belong to the larger field of quantities having sign, but not direction.

P-way Determinants, with an Application to Transvectants.

BY LEPINE HALL RICE.

In this paper an extended definition of a determinant is given which applies to determinants of more than three dimensions, and enables us to remove the restriction in Cayley's law of multiplication and to set up a new case in Scott's law of multiplication. New formulas are obtained for the known process of decomposition of a determinant into determinants of fewer dimensions, and a new process called crossed decomposition is described. Fresh light is thrown upon the function known as a "determinant-permanent," a limitation hitherto thought necessary being done away. Finally a generalization to p dimensions is made of Metzler's theorem in two dimensions concerning a determinant, each of whose elements is the product of k factors.

We lead up to these matters by a brief statement of the elementary theory of 3-way or cubic determinants and permanents.

I. THREE-WAY DETERMINANTS AND PERMANENTS.

1. *Definitions and Fundamental Properties.*—Elements, n^3 in number, can be arranged in a 3-way matrix of order n having n^2 *rows*, n^2 *columns*, and n^2 *normals;* all are called *files*. The matrix divides up into n *strata*, or layers that contain rows and columns and are pierced by normals; and into n *row-normal layers* pierced by columns; and into n *column-normal layers* pierced by rows; all are called *layers*. In triple-index notation, a_{ijk} denotes the element in the i-th stratum, the j-th row-normal layer, and the k-th column-normal layer. To represent 3-way matrices of successive orders, we write:

$$\left\| \begin{array}{cc|cc} a_{111} & a_{112} & a_{211} & a_{212} \\ a_{121} & a_{122} & a_{221} & a_{222} \end{array} \right\|, \quad \left\| \begin{array}{ccc|ccc|ccc} a_{111} & a_{112} & a_{113} & a_{211} & a_{212} & a_{213} & a_{311} & a_{312} & a_{313} \\ a_{121} & a_{122} & a_{123} & a_{221} & a_{222} & a_{223} & a_{321} & a_{322} & a_{323} \\ a_{131} & a_{132} & a_{133} & a_{231} & a_{232} & a_{233} & a_{331} & a_{332} & a_{333} \end{array} \right\|, \text{ etc.}$$

Two or more elements are *conjunctive* if no two of them lie in the same layer of any direction; n conjunctive elements are *perjunctive* and form a *transversal*. When we speak of a determinant of a matrix and use the word "transversal," we mean the product of these elements. The *locant* of an

element is the set of indices which locate it in the matrix. If the locants of a perjunctive set of elements be written in a column, the three subcolumns, of n indices each, are called *ranges*, and the whole is called the locant of the set (transversal); *e. g.*, $a_{122}\, a_{313}\, a_{231}$, ranges 132, 213, 231. The *sign of a range* is $+$ or $-$ according as there is an even or an odd number of inversions of order in the range.

The *determinant* of a 3-way matrix is the algebraic sum of its transversals, each having the sign which is the product of the signs of its second and third ranges.

The *permanent* of a 3-way matrix is the sum of its transversals.

The determinant and permanent being homogeneous linear functions of the elements of any layer, we have obvious theorems as to factors of layers, and as to separation into a sum of 3-way determinants or permanents when there are polynomial elements, and, conversely, as to addition of determinants and of permanents. An interchange of strata in a determinant, or of any parallel layers in a permanent, does not change its value; but an interchange of any two parallel layers other than strata, in a determinant, changes its sign. Hence, if two such layers are alike the determinant vanishes. Hence a multiple of such a layer may be added to any parallel layer without changing the value of the determinant.

A *minor* is formed by striking out an equal number of layers of each direction. It is obvious what we mean by *conjunctive minors* and by *perjunctive minors*. A perjunctive set of minors, formed into a product, with the proper sign, is equal to the sum of a certain number of terms of the determinant; the sign is the sign of that term of the determinant whose elements are the elements in the main diagonals of the minors. In the simple case of an element a_{ijk} and its complementary minor, the sign is $(-1)^{j+k}$. In consequence, the Laplacean expansion of a determinant is formed by partitioning the matrix into two or more sets of parallel layers and forming all possible perjunctive sets of minors occupying the sets of layers, one minor in each set.

2. *Decomposition.*—(i) A 3-way determinant Δ of order n can be decomposed into the sum of $n!$ 2-way determinants whose rows are rows of Δ. For, arranging n perjunctive rows of Δ in the order of the row-normal layers in which they lie, we see that the 2-way determinant

$$\begin{vmatrix} a_{i'11} & a_{i'12} & \dots & a_{i'1n} \\ a_{i''21} & a_{i''22} & \dots & a_{i''2n} \\ \dots & \dots & \dots & \dots \\ a_{i^{(n)}n1} & a_{i^{(n)}n2} & \dots & a_{i^{(n)}nn} \end{vmatrix},$$

whose matrix they form consists of $n!$ terms of Δ, and that the totality of such 2-way determinants consists of all the terms of Δ. They are called *components of* Δ.

(ii) If the rows are arranged in the order of the *strata* in which they lie, then each determinant must have prefixed the sign of the j-range (denoted by $\pm_j$) in the locant of the set of rows.

(iii) (iv) There will be two corresponding forms of decomposition into 2-way determinants whose columns are columns of Δ.

(v) We can also decompose Δ into an algebraic sum of 2-way permanents. Arrange n perjunctive normals in any order, and to the permanent whose matrix they form, prefix the sign of their locant; for, clearly, all of those $n!$ terms of Δ which lie in a perjunctive set of normals have the same sign.

$$(\text{ii})\ \pm_j \begin{vmatrix} a_{1j'1} & a_{1j'2} & \cdots \\ a_{2j''1} & a_{2j''2} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix};\quad (\text{iii}) \begin{vmatrix} a_{i'11} & a_{i''12} & \cdots \\ a_{i'21} & a_{i''22} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix};$$

$$(\text{iv})\ \pm_k \begin{vmatrix} a_{11k'} & a_{21k''} & \cdots \\ a_{12k'} & a_{22k''} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix};\quad (\text{v})\ \pm_j \pm_k \overset{+}{\Bigg|} \begin{matrix} a_{1j'k'} & a_{2j'k'} & \cdots \\ a_{1j''k''} & a_{2j''k''} & \cdots \\ \cdots & \cdots & \cdots \end{matrix} \overset{+}{\Bigg|}^{*}.$$

A 3-way *permanent* may of course be decomposed by rows, columns, or normals, into a sum of 2-way permanents.

3. *Element-Multiplication.*—(i) (Scott's† law of multiplication). The product of two 2-way determinants (or permanents) A and B of order n is expressible as a 3-way determinant (or permanent) C of order n wherein

$$c_{ijk} \equiv a_{ij} b_{ik}.$$

EXAMPLE: $$\overset{+}{\Bigg|} \begin{matrix} a_{11}a_{12} \\ a_{21}a_{22} \end{matrix} \Bigg| \cdot \overset{+}{\Bigg|} \begin{matrix} b_{11}b_{12} \\ b_{21}b_{22} \end{matrix} \Bigg| = \overset{+}{\Bigg|} \begin{matrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{12}b_{11} & a_{12}b_{12} \end{matrix} \Bigg| \begin{matrix} a_{21}b_{21} & a_{21}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{matrix} \Bigg|.$$

From the *prescription* $c_{ijk} \equiv a_{ij} b_{ik}$, whatever be the value of n, we see (i) noting the index i, that columns of A and columns of B are found in normals of C, (ii) noting the index j, that rows of A are found in columns of C and that elements of B run as factors through columns of C; and (iii) noting the index k, that rows of B are found in rows of C, and that elements of A run as factors through rows of C.

---

* "$\overset{+}{|}\ |$" means a determinant or permanent; "$\overset{+}{|}\ \overset{+}{|}$" means a permanent.

† R. F. Scott, "On Cubic Determinants," etc., *Proc. London Math. Soc.*, Vol. XI (1879), p. 17, at p. 23, paragraph 7. In paragraphs 8 and 9, Scott extends the rule so as to give the product of two determinants of p and q dimensions, respectively, in the form of a determinant of $p+q-1$ dimensions. In § 7 of the present paper this rule is extended to determinants as defined in § 5.

This method of examining a prescription will often give at once a good idea of a matrix and may suggest the best way of dealing with it.

In the present case we use decomposition (ii) of the preceding section and find that each component equals B multiplied by a term of A, giving $AB=C$.

(ii) The product of a 2-way determinant A and a 2-way permanent P of order n is expressible as a 3-way determinant C wherein

$$c_{ijk} \equiv a_{jk} p_{ik};$$

that is, a normal of C is a column of P with an a-factor; a column of C is a column of A with a p-factor; and a row of C is a row of A and a row of P. Using decomposition (v), we find that each component equals P multiplied by a term of A, and $AP=C$.

4. *File-Multiplication.*—(Cayley's* law of multiplication.) The product of a 3-way determinant A and a 2-way determinant B of order n is expressible as a 3-way determinant C of order n, wherein

$$c_{ijk} \equiv \sum_{l=1}^{n} a_{ijl} b_{kl};$$

that is, the matrix of each stratum of C is precisely what would result from using the familiar process of multiplying together two 2-way determinants into a 2-way determinant, row into row; one of the determinants being always B, and the other being the determinant of the matrix of the stratum of A corresponding to that of C. In brief, we multiply B into the strata of A to form C.

Using decomposition (i), we find that each component $C_r = A_r B$, whence $C=AB$.

II. Determinants and Permanents of p Dimensions.

5. *Definitions and Fundamental Properties.*—A p-way matrix (or a matrix of *class* p) of order n is formed of n^p elements:

$$\| a_{a_1 a_2 \ldots a_p} \|_n^{(p)}.$$

The matrix can be separated into n layers, in any one of the p directions; a layer is a $(p-1)$-way matrix of n^{p-1} elements. Common to two layers of

* A. Cayley, "On the Theory of Determinants," *Trans. Cambridge Phil. Soc.*, Vol. VIII (1843), p. 75; *Coll. Math. Papers*, Vol. I, p. 63. See §2 of that paper. Cayley extends the rule so as to give the product of two determinants of p and q dimensions, respectively, in the form of a determinant of $p+q-2$ dimensions, but states that the rule is inapplicable when both p and q are odd; a restriction ignored by Scott in paragraph 9 of the paper cited in the first note, and by many others (see "Abrégé de la théorie des déterminants à n dimensions," par Maurice Lecat; Gand, Hoste, 1911, pp. 55 *et seq.*). In §8 of the present paper this rule is extended to determinants as defined in §5, such determinants serving to remove the restriction.

different directions is a $(p-2)$-way *sublayer;* common to three, a $(p-3)$-way sublayer; and so on, until we have, common to $p-1$ layers of different directions, a *file* of n elements, piercing the n parallel layers of the remaining direction. Parallel to this file are $n^{p-1}-1$ other files, together with it containing all the elements of the matrix. Files of the last direction are *rows.*

We speak of *conjunctive* and *perjunctive* elements, files, etc., of *transversals,* and of *locants* and *ranges,* as we did in the special case of a 3-way matrix.

Hitherto a p-way determinant has been defined as the algebraic sum of the transversals of a p-way matrix, the sign of a transversal being determined by arranging its elements in such an order that the values of a fixed index shall read $1, 2, \ldots, n$, and then taking the product of the signs of all the other ranges. It has then been shown that for a matrix of even class the same determinant will result, whatever be the fixed index, but that for a matrix of odd class a different determinant, in general, will result from a different choice of the fixed index. It has also been shown that interchange of layers denoted by the fixed index in a determinant of odd class leaves the value of the determinant unchanged, while the interchange of two layers of any other direction in such a determinant, or the interchange of two layers of any direction in a determinant of even class, changes the sign of the determinant.

It follows that without making the supposition that the elements of a transversal are first arranged in any particular order, we may say that the sign is the product (i) of the signs of all the ranges, in a determinant of even class, and (ii) of the signs of all but one (a fixed one) of the ranges, in a determinant of odd class.

From this point of view, we now generalize the definition of a determinant. We shall call an index (or range, or direction, or file, or set of layers) *signant* or *nonsignant* according as we do or do not take the order therein into account in fixing the sign of a term. In a 2-way determinant both indices are signant; in a 2-way permanent, both indices are nonsignant. In a 3-way determinant, two indices are signant. Passing to matrices of more dimensions than three, we see that it is possible not only to have signant all the indices if the class is even, and all but one if the class is odd, but to have signant a less number in either case, provided only that there be an even number that are signant—two indices in a 4-way or 5-way matrix, two or four indices in a 6-way or 7-way matrix, and so on. We therefore lay down the following:

Definition of a Determinant.—A determinant of a p-way matrix is the sum of all the terms that can be formed by taking a set of perjunctive elements

as factors and prefixing the product of the signs of an even number of chosen ranges.

A permanent might be viewed as one extreme, where the even number is zero; but file-multiplication and dependent processes have no application to permanents, and so we prefer to mention them explicitly when a theorem is true with regard to them, and to understand that a determinant has at least two signant indices.

By a *full-sign* determinant we shall mean a determinant as heretofore defined, with all, or all but one, of the indices signant, according as it is of even or odd class.

If in any determinant two layers of a signant direction be interchanged, the sign of the determinant is changed. Hence, if two such layers are alike, the determinant vanishes. Hence a multiple of one such layer may be added to another without changing the value of the determinant.

Both a determinant and a permanent are of course homogeneous linear functions of the elements of any layer and have the properties resulting.

6. *Decomposition.*—A p-way n-layer determinant Δ can be decomposed into the algebraic sum of $n!$ $(p-1)$-way determinants or permanents, as the case may be. Each of these components has $n!$ components, and so on. Ultimately we arrive at the expression of Δ as the algebraic sum of $(n!)^{p-2}$ 2-way determinants or permanents.

Let the matrix $\|a_{\alpha\beta\dots\kappa\lambda}\|_n^{(p)}$ of Δ be divided up into its $(p-2)$-way sublayers of directions 1 and 2. Denote the sublayer common to the r-th layer of direction 1, and the s-th layer of direction 2 by $a_{rs0\dots0}$. Take n perjunctive sublayers $a_{\alpha'10\dots0}, a_{\alpha''20\dots0}, \dots, a_{\alpha^{(n)}n0\dots0}$ to form a $(p-1)$-way matrix,

$$\left\| \begin{matrix} a_{\alpha'10\dots0} \\ a_{\alpha''20\dots0} \\ \dots\dots \\ a_{\alpha^{(n)}n0\dots0} \end{matrix} \right\|.$$

This is a component of the matrix of Δ; there are $n!$ such components, and all of the locant of an element in Δ except the first index, is the locant of that element in each of the $(n-1)!$ components in which it occurs.

Denoting by the superscripts $\frown$ and $\smile$ the signant and nonsignant indices, and inserting a colon to isolate the locant of a component, we have:

$$\left.\begin{aligned} &(1)\quad |a_{\widehat{\alpha}\widehat{\beta}\dots}|_n^{(p)} = \sum_\alpha \pm_\alpha |a_{\alpha:\breve{\beta}\dots}|_n^{(p-1)}; \\ &(2)\quad |a_{\widehat{\alpha}\breve{\beta}\dots}|_n^{(p)} = \sum_\alpha \pm_\alpha |a_{\alpha:\widehat{\beta}\dots}|_n^{(p-1)}; \\ &(3)\quad |a_{\breve{\alpha}\widehat{\beta}\dots}|_n^{(p)} = \sum_\alpha \quad |a_{\alpha:\widehat{\beta}\dots}|_n^{(p-1)}; \\ &(4)\quad |a_{\breve{\alpha}\breve{\beta}\dots}|_n^{(p)} = \sum_\alpha \quad |a_{\alpha:\breve{\beta}\dots}|_n^{(p-1)}. \end{aligned}\right\} \qquad (D_1)$$

Each index beyond β is signant or nonsignant on the right according as it is signant or nonsignant on the left. In (1) and (2), $\pm_\alpha$ is the sign of the α-range in any transversal when the β-range reads $12\ldots.n$.

Briefly, if α is signant (nonsignant) the components are signed (unsigned) and the signancy of β is reversed (continued).

We see that the components of a determinant may be determinants or may be permanents; but that the components of a permanent must be permanents.

In verifying the formulas we must recall the fact that there are an even number of signant indices. Consider a term of $|a_{\alpha:\overset{\smile}{\beta}\ldots.}|_n^{(p-1)}$ in formula (1). Let its elements be arranged so that the values of β are in the order $12\ldots.n$; then its sign is the product of the signs of the signant ranges beyond β. Prefixing $\pm_\alpha$, we find that we now have the sign proper to this term in $|a_{\overset{\frown}{\alpha}\overset{\frown}{\beta}\ldots.}|_n^{(p)}$. And this sign is preserved when the elements are permuted, provided that we make β nonsignant, because there are an even number of signant ranges beyond β.

If the process be repeated to give the $(p-2)$-way components, we shall have $(n!)^2$ matrices of the form

$$\left\Vert \begin{matrix} a_{\alpha'\beta'10\ldots.0} \\ a_{\alpha''\beta''20\ldots.0} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ a_{\alpha^{(n)}\beta^{(n)}n0\ldots.0} \end{matrix} \right\Vert .$$

To condense the corresponding formulas, we use a double superfix $\asymp$ or $\overset{\frown}{\smile}$, the upper signs to be read together and the lower signs together in each formula:

$$\left.\begin{aligned}
&(1)\quad |a_{\overset{\frown}{\alpha}\overset{\frown}{\beta}\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p)}=\sum_{\alpha,\,\beta}\pm_\alpha\pm_\beta|a_{\alpha\beta:\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p-2)};\\
&(2)\quad |a_{\overset{\frown}{\alpha}\overset{\smile}{\beta}\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p)}=\sum_{\alpha,\,\beta}\pm_\alpha\qquad|a_{\alpha\beta:\overset{\circ}{\gamma}\ldots.}|_n^{(p-2)};\\
&(3)\quad |a_{\overset{\smile}{\alpha}\overset{\frown}{\beta}\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p)}=\sum_{\alpha,\,\beta}\qquad\pm_\beta|a_{\alpha\beta:\overset{\circ}{\gamma}\ldots.}|_n^{(p-2)};\\
&(4)\quad |a_{\overset{\smile}{\alpha}\overset{\smile}{\beta}\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p)}=\sum_{\alpha,\,\beta}\qquad\qquad|a_{\alpha\beta:\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p-2)}.
\end{aligned}\right\}\qquad (D_2)$$

The formulas for *complete decomposition*, that is, separation into 2-way components, are:

$$\left.\begin{aligned}
&(1)\quad |a_{\ldots.\overset{\frown}{\kappa}\overset{\frown}{\lambda}}|_n^{(p)}=\Sigma\pm|a_{\ldots.:\overset{\frown}{\kappa}\overset{\frown}{\lambda}}|_n^{(2)};\\
&(2)\quad |a_{\ldots.\overset{\smile}{\kappa}\overset{\frown}{\lambda}}|_n^{(p)}=\Sigma\pm|a_{\ldots.:\overset{\frown}{\kappa\lambda}}|_n^{(2)};\\
&(3)\quad |a_{\ldots.\overset{\frown}{\kappa}\overset{\smile}{\lambda}}|_n^{(p)}=\Sigma\pm\overset{+}{|}a_{\ldots.:\overset{\smile}{\kappa\lambda}}\overset{+}{|}{}_n^{(2)};\\
&(4)\quad |a_{\ldots.\overset{\smile}{\kappa}\overset{\smile}{\lambda}}|_n^{(p)}=\Sigma\pm\overset{+}{|}a_{\ldots.:\overset{\smile}{\kappa}\overset{\smile}{\lambda}}\overset{+}{|}{}_n^{(2)}.
\end{aligned}\right\}\qquad (D_{p-2})$$

The "$\pm$" is the product of the signs of all the signant ranges before κ.

It will be seen that the 2-way determinants in (1) and (2) and the 2-way permanents in (3) and (4) have for their rows files of Δ of the p-th direction (index λ) *i. e.*, rows of Δ; and that we get determinants when λ is signant, and permanents when λ is nonsignant.

As we can put the indices of any determinant in any order before decomposing it, the formulas are general with respect to such order. This remark applies to some of the later formulas.

In general, if we wish to have r nonsignant indices $\alpha_1\alpha_2\ldots\alpha_r$, and s signant indices $\beta_1\beta_2\ldots\beta_s$ come before the colon, an index γ to come immediately after it (that is, to be the index of the range which is the *base* to which $\pm_{\beta_1}, \pm_{\beta_2}, \ldots, \pm_{\beta_s}$ relate), and wish t nonsignant indices $\delta_1\delta_2\ldots\delta_t$ and u signant indices $\varepsilon_1\varepsilon_2\ldots\varepsilon_u$ to follow γ, the result is:

$$\left.\begin{aligned} &\left| a_{\breve{\alpha}_1\ldots\breve{\alpha}_r\overset{\frown}{\beta}_1\ldots\overset{\frown}{\beta}_s\overset{\smile}{\overset{\frown}{\gamma}}\breve{\delta}_1\ldots\breve{\delta}_t\overset{\frown}{\epsilon}_1\ldots\overset{\frown}{\epsilon}_u} \right|_n^{(r+s+t+u+1)} \\ &\quad = \sum_{\alpha_1\ldots\alpha_r\beta_1\ldots\beta_s} \pm_{\beta_1}\pm_{\beta_2}\ldots\pm_{\beta_s} \left| a_{\alpha_1\ldots\alpha_r\beta_1\ldots\beta_s:\overset{\frown u}{\gamma}\breve{\delta}_1\ldots\breve{\delta}_t\overset{\frown}{\epsilon}_1\ldots\overset{\frown}{\epsilon}_u} \right|_n^{(t+u+1)}, \end{aligned}\right\} \quad (D)$$

where $\frown^u$ is to mean $\frown$ or $\smile$ according as u is odd or even, regardless of whether γ was originally signant or nonsignant. That is, the signancy of γ on the right is to be so taken that there will be an even number of signant indices in the components.

There is an interesting resemblance between the behavior of the signs $\smile$ and $\frown$ and that of $+$ and $-$. See (D_1) and (D_2), where, symbolically, $\frown\frown=\smile$, $\frown\smile=\frown$, $\smile\frown=\frown$, $\smile\smile=\smile$, $\frown\frown\overset{\smile}{\frown}=\overset{\smile}{\frown}$, etc. And, generally, in (D), taking the superfixes of the α's, the β's and γ, we find that $\smile\ldots\smile\frown\ldots\frown\overset{\smile}{\frown}=\overset{\smile}{\frown}$ if s is even, and $=\overset{\frown}{\smile}$ if s is odd.

7. *Element-Multiplication.*—If from the elements of the matrices

$$\| a_{\alpha_1\alpha_2\ldots\alpha_p} \|_n^{(p)}, \quad \| b_{\beta_1\beta_2\ldots\beta_q} \|_n^{(q)}$$

we form a third matrix of class $p+q-1$ and order n, in whose rows the rows of the a-matrix and the rows of the b-matrix are found according to the prescription

$$c_{\alpha_1\ldots\alpha_{p-1}\beta_1\ldots\beta_{q-1}\mu} \equiv a_{\alpha_1\ldots\alpha_{p-1}\mu}\, b_{\beta_1\ldots\beta_{q-1}\mu},$$

then it is seen that any transversal of the c-matrix consists of a transversal of the a-matrix and a transversal of the b-matrix, every possible combination occurring just once.

Let A be either a determinant or the permanent of the a-matrix, and let B be a determinant or the permanent of the b-matrix. Let C be a determinant or the permanent of the c-matrix, according to the result when the signancy of

$\alpha_1 \ldots . \alpha_{p-1}$ in A, and of $\beta_1 \ldots . \beta_{q-1}$ in B is continued in C, and when μ is made nonsignant in C if it is signant or is nonsignant in both A and B, but otherwise is made signant in C:

$$\frown\frown=\smile; \quad \smile\smile=\smile; \quad \smile\frown=\frown; \quad \frown\smile=\frown.$$

Then the evident theorem is:

$$AB=C.$$

The theorem includes as special cases both of the theorems of Section 3.

In the case $\frown\frown=\smile$, if either A or B is of odd class, C has more than one nonsignant index. Thus determinants that are not full-sign determinants not only fit into the cases previously known, but also create a new case.

8. *File-Multiplication.*—Given any two determinants with signant rows:

$$A \equiv \left| a_{\breve{\alpha}_1 \ldots . \breve{\alpha}_f \widehat{\alpha}_{f+1} \ldots . \widehat{\alpha}_p} \right|_n^{(p)}, \quad B \equiv \left| b_{\breve{\beta}_1 \ldots . \breve{\beta}_g \widehat{\beta}_{g+1} \ldots . \widehat{\beta}_q} \right|_n^{(q)},$$

let us compound every row of A into every row of B in the way familiar in the case of 2-way determinants and used in Section 4, so as to form a determinant C of class $p+q-2$ and order n, according to the prescription

$$c_{\breve{\alpha}_1 \ldots . \breve{\alpha}_f \widehat{\alpha}_{f+1} \ldots . \widehat{\alpha}_{p-1} \breve{\beta}_1 \ldots . \breve{\beta}_g \widehat{\beta}_{g+1} \ldots . \widehat{\beta}_{q-1}} \equiv \sum_{\mu=1}^{n} a_{\alpha_1 \ldots . \alpha_{p-1}\mu} b_{\beta_1 \ldots . \beta_{q-1}\mu};$$

that is, combine the locants of the rows of A and B to form the locant of an element of C, and continue the signancy of the indices in those locants. Then

$$AB=C.$$

PROOF: Completely decompose A and B into:

$$\Sigma \pm_{\alpha_{f+1}} \ldots . \pm_{\alpha_{p-2}} \left| a_{\alpha_1 \ldots . \alpha_{p-2} : \widehat{\alpha}_{p-1} \widehat{\alpha}_p} \right|_n^{(2)},$$

$$\Sigma \pm_{\beta_{g+1}} \ldots . \pm_{\beta_{q-2}} \left| b_{\beta_1 \ldots . \beta_{q-2} : \widehat{\beta}_{q-1} \widehat{\beta}_q} \right|_n^{(2)};$$

or,
$$\Sigma \pm_\alpha \left| \begin{matrix} a_{\alpha'_1 \ldots . \alpha'_{p-2} 11} & a_{\alpha'_1 \ldots . \alpha'_{p-2} 12} & \ldots . \\ a_{\alpha''_1 \ldots . \alpha''_{p-2} 21} & a_{\alpha''_1 \ldots . \alpha''_{p-2} 22} & \ldots . \\ \ldots & \ldots & \ldots \end{matrix} \right|_n, \quad \Sigma \pm_\beta \left| \begin{matrix} b_{\beta'_1 \ldots . \beta'_{q-2} 11} & b_{\beta'_1 \ldots . \beta'_{q-2} 12} & \ldots . \\ b_{\beta''_1 \ldots . \beta''_{q-2} 21} & b_{\beta''_1 \ldots . \beta''_{q-2} 22} & \ldots . \\ \ldots & \ldots & \ldots \end{matrix} \right|_n .$$

Multiply rowwise:

$$AB = \Sigma \pm_\alpha \pm_\beta \left| \begin{matrix} \sum_\mu a_{\alpha'_1 \ldots . \alpha'_{p-2} 1\mu} b_{\beta'_1 \ldots . \beta'_{q-2} 1\mu} & \sum_\mu a_{\alpha'_1 \ldots . \alpha'_{p-2} 1\mu} b_{\beta''_1 \ldots . \beta''_{q-2} 2\mu} & \ldots . \\ \sum_\mu a_{\alpha''_1 \ldots . \alpha''_{p-2} 2\mu} b_{\beta'_1 \ldots . \beta'_{q-2} 1\mu} & \sum_\mu a_{\alpha''_1 \ldots . \alpha''_{p-2} 2\mu} b_{\beta''_1 \ldots . \beta''_{q-2} 2\mu} & \ldots . \\ \ldots & \ldots & \ldots \end{matrix} \right|_n$$

$$= \Sigma \pm_\alpha \pm_\beta \left| \begin{matrix} c_{\alpha'_1 \ldots . \alpha'_{p-2} 1 \beta'_1 \ldots . \beta'_{q-2} 1} & c_{\alpha'_1 \ldots . \alpha'_{p-2} 1 \beta''_1 \ldots . \beta''_{q-2} 2} & \ldots . \\ c_{\alpha''_1 \ldots . \alpha''_{p-2} 2 \beta'_1 \ldots . \beta'_{q-2} 1} & c_{\alpha''_1 \ldots . \alpha''_{p-2} 2 \beta''_1 \ldots . \beta''_{q-2} 2} & \ldots . \\ \ldots & \ldots & \ldots \end{matrix} \right|_n .$$

The determinants in this sum are not components of C; but the $n!$ transversals of any one of them are $n!$ of the transversals of C, and the $(n!)^{p+q-4} \cdot n!$

transversals of all are the $(n!)^{p+q-3}$ transversals of C. As to sign, a transversal

$$C\alpha'_1\ldots\alpha'_{p-2}1\beta_1^{(r_1)}\ldots\beta_{q-2}^{(r_1)}r_1\ \ C\alpha''_1\ldots\alpha''_{p-2}2\beta_1^{(r_2)}\ldots\beta_{q-2}^{(r_2)}r_2\ \ldots$$

has the sign $\pm_\alpha \pm_{\beta(r)} \pm_r$ in C, and the sign $\pm_\alpha \pm_\beta \pm_r$ in this sum; and since there are an even number of signant indices in $\beta_1\ldots\beta_{q-2}$, the signs $\pm_{\beta(r)}$ and $\pm_\beta$ are alike. Therefore $AB=C$.

EXAMPLE: $A\equiv|a_{\breve{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3}|_2^{(3)}$, $B\equiv|b_{\breve{\beta}_1\hat{\beta}_2\hat{\beta}_3\hat{\beta}_4\hat{\beta}_5}|_2^{(5)}$;

$$c_{\breve{\alpha}_1\hat{\alpha}_2\breve{\beta}_1\hat{\beta}_2\hat{\beta}_3\hat{\beta}_4}\equiv a_{\alpha_1\alpha_2 1}\,b_{\beta_1\beta_2\beta_3\beta_4 1}+a_{\alpha_1\alpha_2 2}\,b_{\beta_1\beta_2\beta_3\beta_4 2}\,.$$

(It will be convenient here and elsewhere to insert commas in locants to bring out the structure of a matrix; they may be inserted, shifted or removed at pleasure, as they do not change the meaning of a locant.)

$$A=\begin{vmatrix} a_{1,11}a_{1,12} \\ a_{2,21}a_{2,22}\end{vmatrix}+\begin{vmatrix} a_{2,11}a_{2,12} \\ a_{1,21}a_{1,22}\end{vmatrix},\quad B=\begin{vmatrix} b_{111,11}b_{111,12} \\ b_{222,21}b_{222,22}\end{vmatrix}-\begin{vmatrix} b_{112,11}b_{112,12} \\ b_{221,21}b_{221,22}\end{vmatrix}-\ldots\,;$$

$$AB=\begin{vmatrix} a_{11,1}b_{1111,1}+a_{11,2}b_{1111,2} & a_{11,1}b_{2222,1}+a_{11,2}b_{2222,2} \\ a_{22,1}b_{1111,1}+a_{22,2}b_{1111,2} & a_{22,1}b_{2222,1}+a_{22,2}b_{2222,2}\end{vmatrix}$$

$$-\begin{vmatrix} a_{11,1}b_{1121,1}+a_{11,2}b_{1121,2} & a_{11,1}b_{2212,1}+a_{11,2}b_{2212,2} \\ a_{22,1}b_{1121,1}+a_{22,2}b_{1121,2} & a_{22,1}b_{2212,1}+a_{22,2}b_{2212,2}\end{vmatrix}-\ldots$$

$$=\begin{vmatrix} c_{11,1111}c_{11,2222} \\ c_{22,1111}c_{22,2222}\end{vmatrix}-\begin{vmatrix} c_{11,1121}c_{11,2212} \\ c_{22,1121}c_{22,2212}\end{vmatrix}-\ldots=C.$$

This proof depends on the rows of A and B being signant. Apart from the proof, it is plain that one of the two indices $\alpha_p\beta_q$ can not be signant and the other nonsignant, for that would give C an odd number of signant indices. And we proceed to show that both indices can not be nonsignant; from which it follows that a full-sign determinant will not serve to express the file-product of two determinants of odd class—the restriction stated by Cayley in announcing his law of multiplication.

For, take $n=2$, to simplify the statement, In the transversal

$$c_{\alpha'_1\ldots\alpha'_{p-2}1\,\beta'_1\ldots\beta'_{q-2}1}\ \ c_{\alpha''_1\ldots\alpha''_{p-2}2\,\beta''_1\ldots\beta''_{q-2}2}\,,$$

we find the monomial

$$a_{\alpha'_1\ldots\alpha'_{p-2}12}\ b_{\beta'_1\ldots\beta'_{q-2}12}\ a_{\alpha''_1\ldots\alpha''_{p-2}22}\ b_{\beta''_1\ldots\beta''_{q-2}22}\,,$$

which does not consist of transversals of A and B, and we also find this monomial in the transversal

$$c_{\alpha'_1\ldots\alpha'_{p-2}1\,\beta''_1\ldots\beta''_{q-2}2}\ \ c_{\alpha''_1\ldots\alpha''_{p-2}2\,\beta'_1\ldots\beta'_{q-2}1}\,;$$

and these two transversals are affected with the same sign, if α_p and β_q be assumed to be nonsignant, since there are then an even number of signant indices in $\alpha_1 \ldots . \alpha_{p-1}$ and in $\beta_1 \ldots . \beta_{q-1}$. Under this assumption, therefore, $AB \neq C$. The reasoning applies to every order of determinant and to every such monomial; if α_p and β_q in that monomial take one value v_1 times, and another value v_2 times, and so on, with $v_1+v_2+\ldots .=n$, then we shall find that monomial in $v_1! v_2! \ldots .$ transversals having the same sign prefixed.

It is instructive to compare the matrices resulting from element- and file-multiplication. If the polynomial elements of the latter matrix be converted into nonsignant files by deleting the $+$ signs, we have the former matrix, as is shown by the prescriptions.

9. *Crossed Decomposition.*—Let us generalize, in form and in substance, the development of a determinant which is found in the proof of the law of file-multiplication in the previous section. Beginning with the decomposition of any determinant or permanent

$$A \equiv |a_{\alpha_1 \ldots . \alpha_p}|_n^{(p)}$$

into

$$\Sigma \pm |a_{\alpha_1 \ldots . \alpha_{p-2} : \alpha_{p-1} \alpha_p}|_n^{(2)},$$

in each component alike let us cause any set of indices before the colon, provided that among them there are an even number of signant indices or else no signant indices, to take their n sets of values not in the n rows but in the n columns (as in the c-determinants in the above proof). Let us call the new determinants or permanents, with the signs of the components from which they were formed, *crossed components*. Example, $|a_{\breve{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3 \hat{\alpha}_4 \hat{\alpha}_5}|_2^{(5)}$; a component, $\begin{vmatrix} a_{111,11} & a_{111,12} \\ a_{222,21} & a_{222,22} \end{vmatrix}$; the derived crossed component, $\begin{vmatrix} a_{111,11} & a_{122,12} \\ a_{211,21} & a_{222,22} \end{vmatrix}$.

The sum of these crossed components is equal to A. For, let the transposed indices be $\alpha_{h+1} \ldots . \alpha_{p-2}$, and consider any transversal of A:

$$a_{\alpha_1' \ldots . \alpha_{p-2}' 1 \alpha_p'} \, a_{\alpha_1'' \ldots . \alpha_{p-2}'' 2 \alpha_p''} \cdots a_{\alpha_1^{(n)} \ldots . \alpha_{p-2}^{(n)} n \alpha_p^{(n)}}.$$

Let $r_1 r_2 \ldots . r_n$ be such a permutation of $12 \ldots . n$ that

$$\alpha_p^{(r_1)}=1,\ \alpha_p^{(r_2)}=2,\ \ldots .,\ \alpha_p^{(r_n)}=n.$$

This transversal will be found once and only once, in and only in that crossed component whose main diagonal is

$$a_{\alpha_1' \ldots . \alpha_h' \alpha_{h+1}^{(r_1)} \ldots . \alpha_{p-2}^{(r_1)} 11} \, a_{\alpha_1'' \ldots . \alpha_h'' \alpha_{h+1}^{(r_2)} \ldots . \alpha_{p-2}^{(r_2)} 22} \cdots a_{\alpha_1^{(n)} \ldots . \alpha_h^{(n)} \alpha_{h+1}^{(r_n)} \ldots . \alpha_{p-2}^{(r_n)} nn},$$

i. e., in the crossed component

$$\pm \begin{vmatrix} a_{\alpha_1' \ldots . \alpha_h' \alpha_{h+1}^{(r_1)} \ldots . \alpha_{p-2}^{(r_1)} 11} & a_{\alpha_1' \ldots . \alpha_h' \alpha_{h+1}^{(r_2)} \ldots . \alpha_{p-2}^{(r_2)} 12} & \ldots . \\ a_{\alpha_1'' \ldots . \alpha_h'' \alpha_{h+1}^{(r_1)} \ldots . \alpha_{p-2}^{(r_1)} 21} & a_{\alpha_1'' \ldots . \alpha_h'' \alpha_{h+1}^{(r_2)} \ldots . \alpha_{p-2}^{(r_2)} 22} & \ldots . \\ \ldots & \ldots & \ldots \end{vmatrix}_n;$$

and the sign will be right, for if by the symbol

$$\pm\widehat{\alpha}_{h+1\dots p-2}^{(r_1\dots r_n)}$$

we denote the product of the signs of the signant ranges among $\alpha_{h+1}\dots\alpha_{p-2}$ when they take their values in the order indicated by the symbols (r_1), (r_2), $\dots$, (r_n), then we have the equation

$$\pm\widehat{\alpha}_{1\dots h}^{(1\dots n)}\pm\widehat{\alpha}_{h+1\dots p-2}^{(1\dots n)}\pm\alpha_p^{(1\dots n)}=\pm\widehat{\alpha}_{1\dots h}^{(1\dots n)}\pm\widehat{\alpha}_{h+1\dots p-2}^{(r_1\dots r_n)}\pm\alpha_p^{(1\dots n)}.$$

Generalizing in substance, let us decompose any determinant or permanent A into components of class q:

$$A=\Sigma\pm\left|a_{\alpha_{11}\dots\alpha_{1h_1}\alpha_{21}\dots\alpha_{2h_2}\dots\alpha_{q1}\dots\alpha_{qh_q}:\alpha_1\alpha_2\dots\alpha_q}\right|_n^{(q)},$$

the indices before the colon being separated into q groups with an even number of signant indices (0 even) in each group other than the first. Alter each component by causing each group other than the first to take its n sets of values not in the n 1st-way layers, but in the n s-th-way layers for the s-th group. By this alteration the main diagonal is unchanged. Retain the signs of the original components for these *crossed components;* in other words, give each crossed component the sign of its main diagonal term as a term of A.

The sum of these crossed components is equal to A.

EXAMPLE: The determinant $\left|a_{\breve{\alpha}_{11}\alpha_{12}\widehat{\alpha_{21}\alpha_{22}}\widehat{\alpha_{31}\alpha_{32}}\widehat{\alpha_1\alpha_2\alpha_3}}\right|_2^{(9)}$ has 3-way components,

$$\Sigma\pm_{\alpha_{12}}\pm_{\alpha_{21}}\pm_{\alpha_{22}}\pm_{\alpha_{31}}\pm_{\alpha_{32}}\left|a_{\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\alpha_{31}\alpha_{32}:\breve{\alpha}_1\widehat{\alpha_2\alpha_3}}\right|_2^{(8)}.$$

A component: $+\left|\begin{matrix}a_{11,11,11,,111} & a_{11,11,11,,112}\\ a_{11,11,11,,121} & a_{11,11,11,,122}\end{matrix}\middle|\begin{matrix}a_{22,22,22,,211} & a_{22,22,22,,212}\\ a_{22,22,22,,221} & a_{22,22,22,,222}\end{matrix}\right|$; the corresponding crossed component:

$$+\left|\begin{matrix}a_{11,11,11,,111} & a_{11,11,22,,112}\\ a_{11,22,11,,121} & a_{11,22,22,,122}\end{matrix}\middle|\begin{matrix}a_{22,11,11,,211} & a_{22,11,22,,212}\\ a_{22,22,11,,221} & a_{22,22,22,,222}\end{matrix}\right|.$$

PROOF: Let $\prod_{t=1}^{n}a_{\alpha_{11}^{(t)}\dots\alpha_{1h_1}^{(t)}\alpha_{21}^{(t)}\dots\alpha_{2h_2}^{(t)}\dots\alpha_{q1}^{(t)}\dots\alpha_{qh_q}^{(t)}t\alpha_2^{(t)}\dots\alpha_q^{(t)}}$ be any transversal of A. Take $q-1$ permutations of $12\dots n$, viz., $r_{s1}r_{s2}\dots r_{sn}$, $s=2, 3, \dots, q$, such that

$$\alpha_s^{(r_{s1})}=1,\ \alpha_s^{(r_{s2})}=2,\ \dots,\ \alpha_s^{(r_{sn})}=n.$$

The transversal will be found once and only once, in and only in that crossed component whose main diagonal is:

$$\prod_{t=1}^{n}a\left(\alpha_{11}^{(t)}\dots\alpha_{1h_1}^{(t)}\alpha_{21}^{(r_{2t})}\dots\alpha_{2h_2}^{(r_{2t})}\dots\alpha_{q1}^{(r_{qt})}\dots\alpha_{qh_q}^{(r_{qt})}tt\dots t\right).$$

And the sign will be right, since, for each value of s,

$$\pm\widehat{\alpha}_{s1\dots sh_s}^{(1\dots n)}=\pm\widehat{\alpha}_{s1\dots sh_s}^{(r_{s1}\dots r_{sn})}.$$

Thus, in the example above, the transversal

$$a_{112211121}\, a_{221122212}$$

must be found, if at all, in the position

$$\left| \begin{matrix} \\ a_{11,\,22,\,11,,\,121} \end{matrix} \right| \begin{matrix} a_{22,\,11,\,22,,\,212} \\ \\ \end{matrix} \right|$$

in a crossed component; and there is evidently one and only one crossed component in which it is found.

The development in crossed components may be written:

$$A = \sum \left(\prod_s \pm \widehat{\alpha}_{s1 \ldots . sh_s}\right) \left| a \left\{ \begin{array}{c|c} \alpha_{11} \ldots . \alpha_{1h_1} & \alpha_1 \\ \alpha_{21} \ldots . \alpha_{2h_2} & \alpha_2 \\ \ldots\ldots\ldots & \ldots \\ \alpha_{q1} \ldots . \alpha_{qh_q} & \alpha_q \end{array} \right\} \right|_n^{(q)}.$$

10. *Raising and Lowering the Class: The "Determinant-Permanent."*—One may arbitrarily raise the class of a given determinant or permanent by introducing one or more new indices whose values are determined by the values of one or more of the original indices, suitably adjusting their signancy. For example, using Kronecker's symbol $\delta_{i_1 \ldots . i_s} = 1$ if $i_1 = \ldots . = i_s$, otherwise $=0$, we have:

$$|a_{\widehat{\beta}\widehat{\gamma}}|_2^{(2)} = |\delta_{\alpha\beta} a_{\breve{\alpha}\widehat{\beta}\widehat{\gamma}}|_2^{(3)}; \; i.\,e., \; \begin{vmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{vmatrix} = \left| \begin{matrix} a_{11,\,1}\, a_{11,\,2} \\ \cdot \quad \cdot \end{matrix} \right| \begin{matrix} \cdot \quad \cdot \\ a_{22,\,1}\, a_{22,\,2} \end{matrix} \right|.$$

And in general we may introduce as many nonsignant indices as we like, each having any one-to-one correspondence that we like with one of the original indices; and may then arbitrarily render signant any two of the entire set of indices which happen to have a one-to-one correspondence with each other such that both have the same sign.

In particular, if a determinant of even class have two nonsignant indices, there may be introduced a signant index which shall everywhere take the same values as one of the nonsignant indices, the latter being also made signant, and thus being *doubled*, the result being a determinant of odd class with one nonsignant index. *I. e.*,

$$|a_{\breve{\beta}\breve{\gamma}\ldots .}|_n^{(2q)} = |\delta_{\alpha\beta} a_{\widehat{\alpha}\widehat{\beta}\breve{\gamma}\ldots .}|_n^{(2q+1)}.$$

It is by this kind of a determinant of odd class that the product of two full-sign determinants $A^{(p)}$ and $B^{(q)}$ by file-multiplication has heretofore been expressed, when p and q were odd, as a determinant C of class $p+q-1$ (not $p+q-2$); the "fixed index" (nonsignant index) of A has been doubled in C,

and the "fixed index" of B has been made the "fixed index" of C. Of course C would consist largely of zeros.*

On the other hand, let us start with a full-sign determinant A, of odd class, of a type of which C in the last paragraph is a particular case; wherein a group of indices take the same values, another group take the same values, and so on, there being r groups with an even number of indices in each, s groups with an odd number in each, and t single indices, among the latter being the nonsignant index τ_t.† Give to the elements new locants by striking out all but one of each group of indices, put them into a matrix

$$\|a'_{\rho_1\ldots\rho_r\sigma_1\ldots\sigma_s\tau_1\ldots\tau_t}\|_n^{(r+s+t)},$$

and consider the determinant

$$A' \equiv |a'_{\breve{\rho}_1\ldots\breve{\rho}_r\hat{\sigma}_1\ldots\hat{\sigma}_s\hat{\tau}_1\ldots\hat{\tau}_{t-1}\breve{\tau}_t}|_n^{(r+s+t)};$$

noting that $s+t$ is necessarily odd. Evidently,

$$A' = A.$$

Now A' is not a full-sign determinant, but from A' there could be formed a function not involving anything but full-sign determinants and permanents, viz., the function invented by Gegenbauer† and called by him a "determinant-permanent." Decompose A', using (D) of Section 6, into

$$\Sigma \pm_{\sigma_1} \ldots \pm_{\sigma_s} \pm_{\tau_1} \ldots \pm_{\tau_{t-1}} |a'_{\sigma_1\ldots\sigma_s\tau_1\ldots\tau_{t-1}:\breve{\tau}_t\breve{\rho}_1\ldots\breve{\rho}_r}|_n^{(r+1)}.$$

There are $(n!)^{s+t-1}$ of these permanents, and in them the $(\sigma_1\ldots\sigma_s\tau_1\ldots\tau_{t-1})$-ranges are differently written before the τ_t-range, and prefixed to each of them is the sign which is the product of the signs of these ranges. Such being the case, we can arbitrarily construct a determinant B with purely formal elements:

$$B \equiv |b_{\hat{\sigma}_1\ldots\hat{\sigma}_s\hat{\tau}_1\ldots\hat{\tau}_{t-1}\breve{\tau}_t}|_n^{(s+t)},$$

which will have the property that if for each transversal in its expansion in terms we substitute the corresponding permanent, the resulting function will be equal to A. This function is a "determinant-permanent" of class $s+t$ and *genus* $r+1$.

If A be made of *even* class by deleting τ_t, A' will be of class $r+s+t-1$, its components will be of class r, the indices after the colon will be $\breve{\rho}_1\ldots\breve{\rho}_r$, and B will become

$$|b_{\hat{\sigma}_1\ldots\hat{\sigma}_s\hat{\tau}_1\ldots\hat{\tau}_{t-1}\breve{\rho}_1}|_n^{(s+t)};$$

* M. Lecat, "Sur la multiplication des déterminants," *Ann. Soc. Sci. de Bruxelles*, Vol. XXXVII, Part 2, p. 285.

† Abrégé, p. 32.

† L. Gegenbauer, "Einige Sätze über Determinanten höheren Ranges," *Denkschr. Akad. Wien*, Vol. LVII (1890), p. 735. Abrégé (see Note 3), p. 32.

that is, the "determinant-permanent" will be of class $s+t$ and genus r.

In either case the "determinant-permanent" is simply a decomposition of A'.

It has been said that a "determinant-permanent" is necessarily of odd class.* But the decomposition formula (D) shows us that there are two possible decompositions of a determinant $\left| a_{\widehat{\alpha}_1 \cdots \widehat{\alpha}_f \breve{\alpha}_{f+1} \cdots \breve{\alpha}_p} \right|_n^{(p)}$ into a sum of permanents, such as to leave only signant indices before the colon, namely:

$$\Sigma \pm_{\alpha_1} \cdots \pm_{\alpha_f} \left| a_{\alpha_1 \ldots \alpha_f : \breve{\alpha}_{f+1} \cdots \breve{\alpha}_p} \right|_n^{(p-f)},$$
$$\Sigma \pm_{\alpha_1} \cdots \pm_{\alpha_{f-1}} \left| a_{\alpha_1 \ldots \alpha_{f-1} : \breve{\alpha}_f \cdots \breve{\alpha}_p} \right|_n^{(p-f+1)}.$$

Let A', therefore, be decomposed into

$$\Sigma \pm_{\sigma_1} \cdots \pm_{\sigma_s} \pm_{\tau_1} \cdots \pm_{\tau_{t-2}} \left| a'_{\sigma_1 \ldots \sigma_s \tau_1 \ldots \tau_{t-2} : \breve{\tau}_{t-1} (\breve{\tau}_t) \breve{\rho}_1 \cdots \breve{\rho}_r} \right|_n^{(r+1(+1))};$$

then

$$B \equiv \left| b_{\widehat{\sigma}_1 \cdots \widehat{\sigma}_s \widehat{\tau}_1 \cdots \widehat{\tau}_{t-1}} \right|_n^{(s+t-1)},$$

and the "determinant-permanent" is of *even* class.

EXAMPLE:

$$A \equiv \left| a_{\widehat{\alpha_1\beta_2}\widehat{\beta_3\beta_4}\widehat{\gamma_5\gamma_6}\widehat{\epsilon_7\epsilon_8\epsilon_9\epsilon_{10}}} \delta_{\beta_2\beta_3\beta_1} \delta_{\gamma_5\gamma_6} \delta_{\epsilon_7\epsilon_8\epsilon_9\epsilon_{10}} \right|_2^{(10)}.$$
$$A' \equiv \left| a'_{\widehat{\alpha}\widehat{\beta}\breve{\gamma}\breve{\epsilon}} \right|_2^{(4)} = \text{(i)}\, \Sigma \pm_\alpha \pm_\beta \left| a'_{\alpha\beta : \breve{\gamma}\breve{\epsilon}} \right|_2^{(2)} = \text{(ii)}\, \Sigma \pm_\alpha \left| a'_{\alpha : \breve{\beta}\breve{\gamma}\breve{\epsilon}} \right|_2^{(3)}.$$
$$B = \text{(i)} \left| b_{\widehat{\alpha}\widehat{\beta}\breve{\gamma}} \right|_2^{(3)} \cdot B = \text{(ii)} \left| b_{\widehat{\alpha}\widehat{\beta}} \right|_2^{(2)}.$$

(i) Replace $b_{111}b_{222} \quad + b_{112}b_{221} \quad - b_{121}b_{212} \quad - b_{122}b_{211}$

by

$$\begin{vmatrix} a'_{1111} & a'_{1112} \\ a'_{2221} & a'_{2222} \end{vmatrix} + \begin{vmatrix} a'_{1121} & a'_{1122} \\ a'_{2211} & a'_{2212} \end{vmatrix} - \begin{vmatrix} a'_{1211} & a'_{1212} \\ a'_{2121} & a'_{2122} \end{vmatrix} - \begin{vmatrix} a'_{1221} & a'_{1222} \\ a'_{2111} & a'_{2112} \end{vmatrix},$$

that is, by

$$\begin{vmatrix} a_{1,111,11,1111} & a_{1,111,11,2222} \\ a_{2,222,22,1111} & a_{2,222,22,2222} \end{vmatrix} + \cdots.$$

(ii) Replace $b_{11}b_{22} \quad - b_{12}b_{21}$

by

$$\left| \begin{array}{cc|cc} a'_{1111} a'_{1112} & & a'_{2211} a'_{2212} & \\ a'_{1121} a'_{1122} & & a'_{2221} a'_{2222} & \end{array} \right| - \left| \begin{array}{cc|cc} a'_{2111} a'_{2112} & & a'_{1211} a'_{1212} & \\ a'_{2121} a'_{2122} & & a'_{1221} a'_{1222} & \end{array} \right|,$$

and translate into a's as in (i).

In the particular case of the determinant C described in this section, we have, first:

$$\left| a_{\breve{\alpha}_1 \widehat{\alpha}_2 \cdots \widehat{\alpha}_p} \right|_n^{(p)} \cdot \left| b_{\breve{\beta}_1 \widehat{\beta}_2 \cdots \widehat{\beta}_q} \right|_n^{(q)} = \left| c_{\widehat{\alpha_1\alpha_1}\widehat{\alpha}_2 \cdots \widehat{\alpha}_{p-1} \breve{\beta}_1 \widehat{\beta}_2 \cdots \widehat{\beta}_{q-1}} \right|_n^{(p+q-1)}$$
$$= \Sigma \pm_{\alpha_2} \cdots \pm_{\alpha_{p-1}} \pm_{\beta_2} \cdots \pm_{\beta_{q-1}} \left| c'_{\alpha_2 \ldots \alpha_{p-1} \beta_2 \ldots \beta_{q-1} : \breve{\beta}_1 \breve{\alpha}_1} \right|_n^{(2)}, \quad (1)$$

which gives the product of A and B in the form of a "determinant-permanent" of class $p+q-3$ and genus 2, a form previously known. We attach the prime

* Abrégé, p. 32.

to c to give notice that one index of the group $\alpha_1\alpha_1$ has been dropped, just as a' was used above when all but one index in each group had been deleted.

And secondly we have the new form, obtained by writing after the colon any one of the indices that are before it in (1):

$$\Sigma \pm_{\alpha_2} \cdots \pm_{\alpha_{p-1}} \pm_{\beta_2} \cdots \pm_{\beta_{q-2}} \left| c'_{\alpha_2 \ldots \alpha_{p-1} \beta_2 \ldots \beta_{q-2} : \breve{\beta}_{q-1} \breve{\beta}_1 \breve{\alpha}_1} \right|_n^{(3)}, \tag{2}$$

a "determinant-permanent" of even class $p+q-4$ and genus 3.

11. *A Product Determinant.*—A certain example given by Muir was generalized by Metzler in a paper "On a Determinant Each of Whose Elements is the Product of k Factors." * E. H. Moore contributed to the subject † and mentioned that a particular case of the theorem was ascribed to Kronecker. All this work was in two dimensions. Then von Sterneck extended Kronecker's result to p dimensions. ‡ We shall extend Metzler's theorem to p dimensions, including von Sterneck's generalization as a special case.

First let us deal with two sets of determinants, $A^{(1)}, A^{(2)}, \ldots, A^{(l)}$; $B^{(1)}, B^{(2)}, \ldots, B^{(k)}$:

$$A^{(\beta)} \equiv \left| a^{(\beta)}{}_{\breve{\alpha}_1 \ldots \breve{\alpha}_f \hat{\alpha}_{f+1} \ldots \hat{\alpha}_p} \right|_k^{(p)}; \quad B^{(\alpha)} \equiv \left| b^{(\alpha)}{}_{\breve{\beta}_1 \ldots \breve{\beta}_g \hat{\beta}_{g+1} \ldots \hat{\beta}_q} \right|_l^{(q)}.$$

Form a p-way determinant $\bar{A} = A^{(1)} A^{(2)} \ldots A^{(l)}$, of order kl, by placing $A^{(1)}, A^{(2)}, \ldots, A^{(l)}$ along the main diagonal, all other elements being zeros. Using the bipartite signs of order

$$11, 21, \ldots, k1, \quad 12, 22, \ldots, k2, \ldots, \quad 1l, 2l, \ldots, kl,$$

such as Moore employs, we shall have the prescription

$$\bar{a}_{(\breve{\alpha_1\beta})(\breve{\alpha_2\beta}) \ldots (\breve{\alpha_f\beta})(\widehat{\alpha_{f+1}\beta}) \ldots (\widehat{\alpha_p\beta})} \equiv a^{(\beta)}_{\alpha_1 \ldots \alpha_p}.$$

In the same way, form a q-way determinant of order kl, with $B^{(1)}, B^{(2)}, \ldots, B^{(k)}$; then alter its form by a rearrangement of layers, placing the l layers of the first direction containing $B^{(1)}$ in the positions $11, 12, \ldots, 1l$, those containing $B^{(2)}$ in the positions $21, 22, \ldots, 2l$, and so on, and do this for each direction. The resulting determinant $\bar{B} = B^{(1)} B^{(2)} \ldots B^{(k)}$ and the prescription is:

$$\bar{b}_{(\breve{\alpha\beta_1}) \ldots (\breve{\alpha\beta_g})(\widehat{\alpha\beta_{g+1}}) \ldots (\widehat{\alpha\beta_q})} \equiv b^{(\alpha)}_{\beta_1 \ldots \beta_q}.$$

---

* *Am. Math. Monthly*, Vol. VII (1900), p. 151.

† "A Fundamental Remark Concerning Determinantal Notations with the Evaluation of an Important Determinant of Special Form," *Annals of Math.*, Vol. (2) I, p. 177.

‡ R. D. von Sterneck, "Ausdehnung eines Kronecker'schen Satzes auf Determinanten höheren Ranges," *Rend. Palermo*, Vol. XXX (1910), p. 58. Lecat points out the fact that von Sterneck's theorem does not hold when the classes are both odd; Abrégé. p. 63. The present extension is not subject to that restriction.

Multiply together $\bar{A}$ and $\bar{B}$ by rows into a determinant U of class $p+q-2$ and order kl. The elements of U which are not zeros are monomials, since a row of $\bar{A}$ that is not blank contains nonzero elements only in the β-th set of k places, while a row of $\bar{B}$ that is not blank contains only one nonzero element in each set of k places. The a-element and b-element whose product is a u-element will be the a-element in whose locant $\alpha_p=\alpha$ and the b-element in whose locant $\beta_q=\beta$; we therefore change α to α_p and β to β_q to form the resulting prescription:

$$u_{(\breve{\alpha}_1\beta_q)\cdots\cdots(\breve{\alpha}_f\beta_q)(\widehat{\alpha}_{f+1}\beta_q)\cdots\cdots(\widehat{\alpha}_{p-1}\beta_q)(\breve{\alpha}_p\beta_1)\cdots\cdots(\breve{\alpha}_p\beta_g)(\widehat{\alpha}_p\beta_{g+1})\cdots\cdots(\widehat{\alpha}_p\beta_{q-1})}\equiv a^{(\beta_q)}_{\alpha_1\ldots\alpha_p}\, b^{(\alpha_p)}_{\beta_1\ldots\beta_q},$$

where at least $(\alpha_{p-1}\beta_q)$ and $(\alpha_p\beta_{q-1})$ are signant.

This gives the theorem

$$U=A^{(1)}\ldots A^{(l)}B^{(1)}\ldots B^{(k)}.$$

For 2-way determinants the prescription becomes

$$u_{(\widehat{\alpha}_1\beta_2)(\widehat{\alpha}_2\beta_1)}\equiv a^{(\beta_2)}_{\alpha_1\alpha_2}\, b^{(\alpha_2)}_{\beta_1\beta_2},$$

which agrees with the theorem designated T_2 by Moore.

If $A^{(1)}=A^{(2)}=\ldots=A^{(l)}=A$, say, and $B^{(1)}=B^{(2)}=\ldots=B^{(k)}=B$, we have:

$$U=A^lB^k,$$

which is an extension of von Sterneck's theorem to less than full-sign determinants.

EXAMPLE: $p=3$, $q=3$, $k=2$, $l=2$. $A'\equiv|a'_{\breve{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_3}|^{(3)}_2$, $A''\equiv|a''_{\breve{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_3}|^{(3)}_2$, $B'\equiv|b'_{\breve{\beta}_1\widehat{\beta}_2\widehat{\beta}_3}|^{(3)}_2$, $B''\equiv|b''_{\breve{\beta}_1\widehat{\beta}_2\widehat{\beta}_3}|^{(3)}_2$. For $\bar{A}$ and $\bar{B}$ we have:

$$\begin{array}{c|cccc|cccc|cccc|cccc|}
\multicolumn{1}{c}{} & \multicolumn{4}{c}{(11)} & \multicolumn{4}{c}{(21)} & \multicolumn{4}{c}{(12)} & \multicolumn{4}{c}{(22)} \\
\multicolumn{1}{c}{} & (11) & (21) & (12) & \multicolumn{1}{c}{(22)} & (11) & (21) & (12) & \multicolumn{1}{c}{(22)} & (11) & (21) & (12) & \multicolumn{1}{c}{(22)} & (11) & (21) & (12) & \multicolumn{1}{c}{(22)} \\
(11) & a'_{111} & a'_{112} & \cdot & \cdot & a'_{211} & a'_{212} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(21) & a'_{121} & a'_{122} & \cdot & \cdot & a'_{221} & a'_{222} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(12) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a''_{111} & a''_{112} & \cdot & \cdot & a''_{211} & a''_{212} \\
(22) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a''_{121} & a''_{122} & \cdot & \cdot & a''_{221} & a''_{222}
\end{array};$$

$$\begin{array}{c|cccc|cccc|cccc|cccc|}
(11) & b'_{111} & \cdot & b'_{112} & \cdot & \cdot & \cdot & \cdot & \cdot & b'_{211} & \cdot & b'_{212} & \cdot & \cdot & \cdot & \cdot & \cdot \\
(21) & \cdot & \cdot & \cdot & \cdot & \cdot & b''_{111} & \cdot & b''_{112} & \cdot & \cdot & \cdot & \cdot & \cdot & b''_{211} & \cdot & b''_{212} \\
(12) & b'_{121} & \cdot & b'_{122} & \cdot & \cdot & \cdot & \cdot & \cdot & b'_{221} & \cdot & b'_{222} & \cdot & \cdot & \cdot & \cdot & \cdot \\
(22) & \cdot & \cdot & \cdot & \cdot & \cdot & b''_{121} & \cdot & b''_{122} & \cdot & \cdot & \cdot & \cdot & \cdot & b''_{221} & \cdot & b''_{222}
\end{array}.$$

The prescription is:

$$u_{(\breve{\alpha}_1\beta_3)(\widehat{\alpha}_2\beta_3)(\breve{\alpha}_3\beta_1)(\widehat{\alpha}_3\beta_2)}\equiv a^{(\beta_3)}_{\alpha_1\alpha_2\alpha_3}\, b^{(\alpha_3)}_{\beta_1\beta_2\beta_3}.$$

We shall condense the representation of U by writing the b-factor of each element under the a-factor. It is convenient to have the four values of the first index of u appear in the four large horizontal subdivisions; those of the second in the large vertical subdivisions; and those of the third and fourth in the horizontal and vertical lines in each square common to two intersecting subdivisions. Thus one of the sixteen rows of $\bar{A}$ is associated with each of the sixteen squares, while one of the sixteen rows of $\bar{B}$ is associated with each of the sixteen places in each square (the same place in every square):

		(11)				(21)				(12)				(22)			
		(11)	(21)	(12)	(22)	(11)	(21)	(12)	(22)	(11)	(21)	(12)	(22)	(11)	(21)	(12)	(22)
(11)	(11)	$\genfrac{}{}{0pt}{}{a'_{111}}{b'_{111}}$	·	$\genfrac{}{}{0pt}{}{a'_{111}}{b'_{121}}$	·	$\genfrac{}{}{0pt}{}{a'_{121}}{b'_{111}}$	·	$\genfrac{}{}{0pt}{}{a'_{121}}{b'_{121}}$	·	·	·	·	·	·	·	·	·
	(21)	·	$\genfrac{}{}{0pt}{}{a'_{112}}{b''_{111}}$	·	$\genfrac{}{}{0pt}{}{a'_{112}}{b''_{121}}$	·	$\genfrac{}{}{0pt}{}{a'_{122}}{b''_{111}}$	·	$\genfrac{}{}{0pt}{}{a'_{122}}{b''_{121}}$	·	·	·	·	·	·	·	·
	(12)	$\genfrac{}{}{0pt}{}{a'_{111}}{b'_{211}}$	·	$\genfrac{}{}{0pt}{}{a'_{111}}{b'_{221}}$	·	$\genfrac{}{}{0pt}{}{a'_{121}}{b'_{211}}$	·	$\genfrac{}{}{0pt}{}{a'_{121}}{b'_{221}}$	·	·	·	·	·	·	·	·	·
	(22)	·	$\genfrac{}{}{0pt}{}{a'_{112}}{b''_{211}}$	·	$\genfrac{}{}{0pt}{}{a'_{112}}{b''_{221}}$	·	$\genfrac{}{}{0pt}{}{a'_{122}}{b''_{211}}$	·	$\genfrac{}{}{0pt}{}{a'_{122}}{b''_{221}}$	·	·	·	·	·	·	·	·
(21)	(11)	$\genfrac{}{}{0pt}{}{a'_{211}}{b'_{111}}$	·	$\genfrac{}{}{0pt}{}{a'_{211}}{b'_{121}}$	·	$\genfrac{}{}{0pt}{}{a'_{221}}{b'_{111}}$	·	$\genfrac{}{}{0pt}{}{a'_{221}}{b'_{121}}$	·	·	·	·	·	·	·	·	·
	(21)	·	$\genfrac{}{}{0pt}{}{a'_{212}}{b''_{111}}$	·	$\genfrac{}{}{0pt}{}{a'_{212}}{b''_{121}}$	·	$\genfrac{}{}{0pt}{}{a'_{222}}{b''_{111}}$	·	$\genfrac{}{}{0pt}{}{a'_{222}}{b''_{121}}$	·	·	·	·	·	·	·	·
	(12)	$\genfrac{}{}{0pt}{}{a'_{211}}{b'_{211}}$	·	$\genfrac{}{}{0pt}{}{a'_{211}}{b'_{221}}$	·	$\genfrac{}{}{0pt}{}{a'_{221}}{b'_{211}}$	·	$\genfrac{}{}{0pt}{}{a'_{221}}{b'_{221}}$	·	·	·	·	·	·	·	·	·
	(22)	·	$\genfrac{}{}{0pt}{}{a'_{212}}{b''_{211}}$	·	$\genfrac{}{}{0pt}{}{a'_{212}}{b''_{221}}$	·	$\genfrac{}{}{0pt}{}{a'_{222}}{b''_{211}}$	·	$\genfrac{}{}{0pt}{}{a'_{222}}{b''_{221}}$	·	·	·	·	·	·	·	·
(12)	(11)	·	·	·	·	·	·	·	·	$\genfrac{}{}{0pt}{}{a''_{111}}{b'_{112}}$	·	$\genfrac{}{}{0pt}{}{a''_{111}}{b'_{122}}$	·	$\genfrac{}{}{0pt}{}{a''_{121}}{b'_{112}}$	·	$\genfrac{}{}{0pt}{}{a''_{121}}{b'_{122}}$	·
	(21)	·	·	·	·	·	·	·	·	·	$\genfrac{}{}{0pt}{}{a''_{112}}{b''_{112}}$	·	$\genfrac{}{}{0pt}{}{a''_{112}}{b''_{122}}$	·	$\genfrac{}{}{0pt}{}{a''_{122}}{b''_{112}}$	·	$\genfrac{}{}{0pt}{}{a''_{122}}{b''_{122}}$
	(12)	·	·	·	·	·	·	·	·	$\genfrac{}{}{0pt}{}{a''_{111}}{b'_{212}}$	·	$\genfrac{}{}{0pt}{}{a''_{111}}{b'_{222}}$	·	$\genfrac{}{}{0pt}{}{a''_{121}}{b'_{212}}$	·	$\genfrac{}{}{0pt}{}{a''_{121}}{b'_{222}}$	·
	(22)	·	·	·	·	·	·	·	·	·	$\genfrac{}{}{0pt}{}{a''_{112}}{b''_{212}}$	·	$\genfrac{}{}{0pt}{}{a''_{112}}{b''_{222}}$	·	$\genfrac{}{}{0pt}{}{a''_{122}}{b''_{212}}$	·	$\genfrac{}{}{0pt}{}{a''_{122}}{b''_{222}}$
(22)	(11)	·	·	·	·	·	·	·	·	$\genfrac{}{}{0pt}{}{a''_{211}}{b'_{112}}$	·	$\genfrac{}{}{0pt}{}{a''_{211}}{b'_{122}}$	·	$\genfrac{}{}{0pt}{}{a''_{221}}{b'_{112}}$	·	$\genfrac{}{}{0pt}{}{a''_{221}}{b'_{122}}$	·
	(21)	·	·	·	·	·	·	·	·	·	$\genfrac{}{}{0pt}{}{a''_{212}}{b''_{112}}$	·	$\genfrac{}{}{0pt}{}{a''_{212}}{b''_{122}}$	·	$\genfrac{}{}{0pt}{}{a''_{222}}{b''_{112}}$	·	$\genfrac{}{}{0pt}{}{a''_{222}}{b''_{122}}$
	(12)	·	·	·	·	·	·	·	·	$\genfrac{}{}{0pt}{}{a''_{211}}{b'_{212}}$	·	$\genfrac{}{}{0pt}{}{a''_{211}}{b'_{222}}$	·	$\genfrac{}{}{0pt}{}{a''_{221}}{b'_{212}}$	·	$\genfrac{}{}{0pt}{}{a''_{221}}{b'_{222}}$	·
	(22)	·	·	·	·	·	·	·	·	·	$\genfrac{}{}{0pt}{}{a''_{212}}{b''_{212}}$	·	$\genfrac{}{}{0pt}{}{a''_{212}}{b''_{222}}$	·	$\genfrac{}{}{0pt}{}{a''_{222}}{b''_{212}}$	·	$\genfrac{}{}{0pt}{}{a''_{222}}{b''_{222}}$

We deal next with three sets of determinants:

$$A^{(11)},\ A^{(21)},\ \ldots,\ A^{(l1)},\ A^{(12)},\ \ldots,\ A^{(l2)},\ \ldots,\ A^{(1m)},\ \ldots,\ A^{(lm)},$$
$$B^{(11)},\ B^{(21)},\ \ldots,\ B^{(k1)},\ B^{(12)},\ \ldots,\ B^{(k2)},\ \ldots,\ B^{(1m)},\ \ldots,\ B^{(km)},$$
$$C^{(11)},\ C^{(21)},\ \ldots,\ C^{(k1)},\ C^{(12)},\ \ldots,\ C^{(k2)},\ \ldots,\ C^{(1l)},\ \ldots,\ C^{(kl)};$$

where

$$C^{(\alpha\beta)} \equiv \left| c^{(\alpha\beta)}_{\breve{\gamma}_1 \ldots \breve{\gamma}_h \hat{\gamma}_{h+1} \ldots \hat{\gamma}_r} \right|^{(r)}_{m}$$

Forming now $U^{(1)} = A^{(11)} A^{(21)} \ldots A^{(l1)} B^{(11)} B^{(21)} \ldots B^{(k1)}$, and so on up to $U^{(m)} = A^{(1m)} \ldots B^{(km)}$, construct $\overline{U} = U^{(1)} \ldots U^{(m)}$ (as we did $\overline{A}$) so that

$$\overline{u}_{\overset{\smile}{(\alpha_1\beta_q\gamma)} \ldots \overset{\smile}{(\alpha_f\beta_q\gamma)} \overset{\frown}{(\alpha_{f+1}\beta_q\gamma)} \ldots \overset{\frown}{(\alpha_{p-1}\beta_q\gamma)} \overset{\smile}{(\alpha_p\beta_1\gamma)} \ldots \overset{\smile}{(\alpha_p\beta_g\gamma)} \overset{\frown}{(\alpha_p\beta_{g+1}\gamma)} \ldots \overset{\frown}{(\alpha_p\beta_{q-1}\gamma)}} \equiv a^{(\beta_q\gamma)}_{\alpha_1 \ldots \alpha_p}\, b^{(\alpha_p\gamma)}_{\beta_1 \ldots \beta_q};$$

and construct $\overline{C}$ (as we did $\overline{B}$) so that

$$\overline{c}_{\overset{\smile}{(\alpha\beta\gamma_1)} \ldots \overset{\smile}{(\alpha\beta\gamma_h)} \overset{\frown}{(\alpha\beta\gamma_{h+1})} \ldots \overset{\frown}{(\alpha\beta\gamma_r)}} \equiv c^{(\alpha\beta)}_{\gamma_1 \ldots \gamma_r}.$$

Multiply together $\overline{U}$ and $\overline{C}$ by rows into a determinant V of class $p+q+r-4$ and order klm. The elements of V which are not zeros are monomials, and $\alpha_p = \alpha$, $\beta_{q-1} = \beta$, $\gamma_r = \gamma$. Detaching the suffixes of α, β and γ in the locant of an element of V, we have the prescription:

$$v \left\{ \begin{array}{lllllll} \alpha: & \breve{1} \ldots \breve{f} & \widehat{f+1} \ldots \widehat{p-1} & \breve{p} \ldots \breve{p} & \hat{p} \ldots \hat{p} & \breve{p} \ldots \breve{p} & \hat{p} \ldots \hat{p} \\ \beta: & q \ldots q & q \ldots q & 1 \ldots g & g+1 \ldots q-2 & q-1 \ldots q-1 & q-1 \ldots q-1 \\ \gamma: & r \ldots r & r \ldots r & r \ldots r & r \ldots r & 1 \ldots h & h+1 \ldots r-1 \end{array} \right\}$$
$$\equiv a^{(\beta_q\gamma_r)}_{\alpha_1 \ldots \alpha_p}\, b^{(\gamma_r\alpha_p)}_{\beta_1 \ldots \beta_q}\, c^{(\alpha_p\beta_{q-1})}_{\gamma_1 \ldots \gamma_r}.$$

Here at least the indices $(\alpha_{p-1}\beta_q\gamma_r)$ and $(\alpha_p\beta_{q-1}\gamma_{r-1})$ are signant. The superfixes on the right are the sets of two consecutive indices in the sequence

$$\beta_q\, \gamma_r\, \alpha_p\, \beta_{q-1}.$$

The prescription applicable to *four* sets of determinants is obtainable by: (i) changing v to w; (ii) subjoining to the locant the line

$$\delta:\ s \ldots s \ldots s \ldots s\ 1 \ldots i\ i+1 \ldots s-1,$$

the last s falling under γ_{r-2}, the 1 under γ_{r-1}, and the values p, $q-1$, and $r-1$ being continued over $\delta_2 \ldots \delta_{s-1}$; (iii) annexing on the right

$$d^{(\alpha_p\beta_{q-1}\gamma_{r-1})}_{\delta_1 \ldots \delta_s};$$

and (iv) inserting δ_s after γ_r and before α_p in the first three superfixes, so that the superfixes are now the sets of three consecutive indices in the sequence

$$\beta_q\, \gamma_r\, \delta_s\, \alpha_p\, \beta_{q-1}\, \gamma_{r-1}.$$

Each prescription is obtained in like manner from the prescription that precedes, and we have the following general theorem:

The product of r sets of determinants

$$A^{(hg_h)} \equiv \left| a^{(hg_h)}_{\smile\alpha_{h1} \smile\alpha_{h2} \ldots \smile\alpha_{hf_h} \frown\alpha_{hf_h+1} \ldots \frown\alpha_{hp_h}} \right|^{(p_h)}_{n_h},$$

where $h=1, 2, \ldots, r$, *and where* g_h *is the h-th set of* $r-1$ *consecutive indices in the sequence*

$$\alpha_{2p_2}\alpha_{3p_3} \ldots \alpha_{rp_r}\alpha_{1p_1}\alpha_{2,p_2-1}\alpha_{3,p_3-1} \ldots \alpha_{r-1,p_{r-1}-1},$$

is expressible as a determinant A *of class* $p_1+\ldots+p_r-2(r-1)$ *and order* $n_1n_2\ldots n_r$, *in which all elements are zeros excepting those given by the prescription*

$$a\left\{\begin{array}{l} (\overset{\smile}{\alpha_{11}g_1}) \ldots (\overset{\smile}{\alpha_{1f_1}g_1})(\overset{\frown}{\alpha_{1f_1+1}g_1}) \ldots (\overset{\frown}{\alpha_{1p_1-1}g_1}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\overset{\smile}{\alpha_{k1}g_k}) \ldots (\overset{\smile}{\alpha_{kf_k}g_k})(\overset{\frown}{\alpha_{kf_k+1}g_k}) \ldots (\overset{\frown}{\alpha_{kp_k-2}g_k}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\overset{\smile}{\alpha_{r1}g_r}) \ldots (\overset{\smile}{\alpha_{rf_r}g_r})(\overset{\frown}{\alpha_{rf_r+1}g_r}) \ldots (\overset{\frown}{\alpha_{rp_r-1}g_r}) \end{array}\right\} \equiv \prod_{h=1}^{r} a^{(hg_h)}_{\alpha_{h1}\ldots\alpha_{hp_h}},$$

where $k=2, 3, \ldots, r-1$.

Remark that at least the indices $(\alpha_{1p_1-1}g_1)$ and $(\alpha_{rp_r-1}g_r)$ are signant.

If all the determinants are of two dimensions, the prescription takes the form

$$a_{(\alpha_{11}\alpha_{22}\alpha_{32}\ldots\alpha_{r2})(\alpha_{12}\alpha_{21}\alpha_{31}\ldots\alpha_{r-1,1}\alpha_{r1})} \equiv \prod_{h=1}^{r} a^{(hg_h)}_{\alpha_{h1}\alpha_{h2}},$$

the sequence to which g_h applies being

$$\alpha_{22}\alpha_{32} \ldots \alpha_{r2}\alpha_{12}\alpha_{21}\alpha_{31} \ldots \alpha_{r-1,1}.$$

This agrees with the theorem of Metzler designated T_s by Moore.

12. *An Application to Transvectants.**

THEOREM: *If the* n^{p+q} *k-th transvectants of all possible pairs of binary forms taken from two sets*

$$\|f_{\alpha_1\ldots\alpha_p}\|^{(p)}_n, \quad \|\phi_{\beta_1\ldots\beta_q}\|^{(q)}_n, \quad n \geqq k+2,$$

be made the elements of a determinant

$$C \equiv \left| c_{\smile\alpha_1 \ldots \smile\alpha_g \frown\alpha_{g+1} \ldots \frown\alpha_p \smile\beta_1 \ldots \smile\beta_h \frown\beta_{h+1} \ldots \frown\beta_q} \right|^{(p+q)}_n, \quad \begin{cases} p-g \text{ odd}, \\ q-h \text{ odd}, \end{cases}$$

with $$c_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_q} \equiv (f_{\alpha_1\ldots\alpha_p}, \phi_{\beta_1\ldots\beta_q})^k,$$

then $$C \equiv 0.$$

---

*Special cases of this theorem have been given by L. Gegenbauer (*loc. cit.*); see Abrégé, p. 94, where an important restriction on Gegenbauer's results, not mentioned by him, is brought out.

PROOF: We can form two null determinants A and B, whose elements are k-th derivatives of the f's and ϕ's respectively, with suitable numerical factors, such that $A \cdot B = C$. Letting $m_{a_1 \ldots a_p}$, $\mu_{\beta_1 \ldots \beta_q}$ be the degrees of $f_{a_1 \ldots a_p}$, $\phi_{\beta_1 \ldots \beta_q}$, respectively, set up:

$$A \equiv \left| a_{\breve{a}_1 \ldots \breve{a}_\rho \hat{a}_{\rho+1} \ldots \hat{a}_p \hat{\lambda}} \right|_n^{(p+1)}, \quad a_{a_1 \ldots a_p \lambda} \equiv \frac{(m_{a_1 \ldots a_p} - k)\,!}{m_{a_1 \ldots a_p}\,!} \cdot \frac{\partial^k f_{a_1 \ldots a_p}}{\partial_{x_1}^{k-\lambda+1} \partial_{x_2}^{\lambda-1}};$$

$$B \equiv \left| b_{\breve{\beta}_1 \ldots \breve{\beta}_h \hat{\beta}_{h+1} \ldots \hat{\beta}_q \hat{\lambda}} \right|_n^{(q+1)}, \quad b_{\beta_1 \ldots \beta_q \lambda} \equiv (-1)^{\lambda-1} \binom{k}{\lambda-1} \frac{(\mu_{\beta_1 \ldots \beta_q} - k)\,!}{\mu_{\beta_1 \ldots \beta_q}\,!} \cdot \frac{\partial^k \phi_{\beta_1 \ldots \beta_q}}{\partial_{x_1}^{\lambda-1} \partial_{x_2}^{k-\lambda+1}}.$$

These prescriptions fill only the first $k+1$ places in each row (file of the last direction), and we shall fill the remaining places in each row with zeros, the result being that one or more layers in A and in B will consist of zeros, whence $A=0$ and $B=0$. Obviously $A \cdot B = C$, the multiplication being of row into row.

COROLLARY 1. *If the n^4 k-th transvectants of all possible pairs of binary forms taken from two sets*

$$\begin{Vmatrix} f_{11} \cdots f_{1n} \\ \cdots\cdots \\ f_{n1} \cdots f_{nn} \end{Vmatrix}, \quad \begin{Vmatrix} \phi_{11} \cdots \phi_{1n} \\ \cdots\cdots \\ \phi_{n1} \cdots \phi_{nn} \end{Vmatrix}, \quad n \geqq k+2.$$

be made the elements of a 4-way determinant with two signant indices:

$$C \equiv \left| c_{\breve{a}_1 \hat{a}_2 \breve{\beta}_1 \hat{\beta}_2} \right|_n^{(4)}, \quad c_{a_1 a_2 \beta_1 \beta_2} \equiv (f_{a_1 a_2}, \phi_{\beta_1 \beta_2})^k,$$

then $$C \equiv 0.$$

COROLLARY 2. *If the n^2 k-th transvectants of all possible pairs of binary forms taken from two sets*

$$f_1, f_2, \ldots, f_n, \quad \phi_1, \phi_2, \ldots, \phi_n, \quad n \geqq k+2,$$

be made the elements of a 2-way determinant

$$C \equiv \begin{vmatrix} (f_1, \phi_1)^k \cdots (f_1, \phi_n)^k \\ \cdots\cdots\cdots \\ (f_n, \phi_1)^k \cdots (f_n, \phi_n)^k \end{vmatrix}_n,$$

then $$C \equiv 0.$$

This corollary includes as a special case Gordan's result:

$$\begin{vmatrix} (f_1, \phi_1)^2 \cdots (f_1, \phi_4)^2 \\ \cdots\cdots\cdots \\ (f_4, \phi_1)^2 \cdots (f_4, \phi_4)^2 \end{vmatrix}_4 \equiv 0.$$

CAMBRIDGE, MASSACHUSETTS.

*On a Certain General Class of Functional Equations.**

By W. Harold Wilson.

§ 1. *Introduction and General Considerations.*

Addition formulae of the general type

$$G[f(x+y),\, f(x),\, f(y)]=0,$$

where G is a polynomial in its three arguments, play a prominent rôle in the theory of elliptic functions. A natural generalization of such formulae is

$$P[x,\, y,\, f(x),\, f(y),\, f(\alpha_1 x+\beta_1 y),\, \ldots,\, f(\alpha_n x+\beta_n y)]=0, \tag{I}$$

where

(i) P denotes a polynomial in its $n+4$ arguments such that every argument involving f is explicitly present;

(ii) x and y are independent variables;

(iii) α_i and β_i, $i=1, 2, \ldots, n$, are given constants;† and,

(iv) $f(x)$ is an unknown single-valued function to be determined so that equation (I) shall be identically satisfied.‡

Equation (I) *is said to be of order* n. *The degree* m *of* P *in the function* f *is said to be the degree of equation* (I).

---

* Read before the American Mathematical Society (at Chicago), April 6, 1917.

† For the purposes of this paper it is convenient to carry certain hypotheses in regard to the α's and β's. A statement of these hypotheses is to be found below.

‡ A theorem of some interest in the general theory of these functional equations is that *every solution $f(x)$ of equation* (I) *is a solution of a similar equation in which x and y occur only in the arguments of the function f.* To prove this arrange P as a polynomial in x and y. The substitutions $x=s+k_i t$, $y=t$, where $k_0=0$ and

$$k_i \neq k_\lambda \pm \frac{\beta_h}{\alpha_h}, \quad k_i \neq k_\lambda + \frac{\beta_h}{\alpha_h} - \frac{\beta_j}{\alpha_j},$$

$\lambda=0, 1, \ldots, i-1$, $h, j=1, 2, \ldots, n$, transform (I) into equations similar to (I) such that the highest degree in s and t is the same for all of them. It is easily seen that a finite number of non-zero k's may be employed such that the variables s and t may be eliminated from these equations, in so far as they occur as coefficients, by Sylvester's dialytic method of elimination. The result of this elimination is an equation (II) which states that a polynomial Q in the function f has the value zero. The arguments of f are linear combinations of two independent variables s and t. Hence (II) is similar to (I), although in general its order and degree will differ from those of (I). Furthermore, if no two arguments of f in (I) are proportional, then no two arguments of f in (II) are proportional.

Cauchy* discussed two special cases of (I) and two related equations, namely:

$$f(x+y)=f(x)+f(y),\quad f(x+y)=f(x)f(y),$$
$$f(xy)=f(x)+f(y),\quad f(xy)=f(x)f(y).$$

One or the other of the first two of these equations has since been treated† by Darboux, E. B. Wilson, Vallée Poussin, Schimmack and Hamel. Carmichael‡ has given a generalization of the Cauchy equations while Jensen§ has discussed several applications of them. Cauchy‖ has treated the equation

$$\phi(x+y)+\phi(x-y)=2\phi(x)\phi(y).$$

Carmichael¶ has considered the equations

$$h(x+y)h(x-y)=h^2(x)+h^2(y)-c^2,\quad g(x+y)g(x-y)=g^2(x)-g^2(y).$$

Van Vleck and H'Doubler** have discussed the equation

$$\psi(x+y)\psi(x-y)=[\psi(x)\psi(y)]^2.$$

Other related equations have been considered by several writers, and systems of functional equations have also been treated.

It seems that no systematic account of a general theory for equations of the form (I) has ever been undertaken. This paper is designed to contribute to such an account. The equations considered in the principal part (§§ 2 to 8) of the paper are linear homogeneous equations with constant coefficients. They may be written in the form

$$\sum_{i=1}^{n}\gamma_i f(\alpha_i x+\beta_i y)+\gamma_{n+1}f(x)+\gamma_{n+2}f(y)=0. \tag{1}$$

It will be shown that if some α's and β's having different subscripts are zero and no ratio α_i/β_i of non-zero α's and β's is distinct from all the remaining ratios, the equation is exceptional. The exceptional case receives mention only in §§ 6 and 11. There is no loss of generality in assuming that no α is zero in the non-exceptional case. For convenience in exposition the hypothesis will be carried in the text that in addition to no α being zero, no β is zero, and no two ratios α_i/β_i are equal. The additional argumentation for the remaining non-exceptional cases is supplied in footnotes.

* *Cours d'Analyse* (1821), Chapter 5. Cauchy treated the last two equations by transforming them into the first two equations. It is obvious that similar transformations may be applied to reduce more general equations to the form of those considered in this paper.

† Darboux, *Mathematische Annalen*, Vol. XVII (1880), p. 56. E. B. Wilson, *Annals of Mathematics*, Vol. I, Ser. 2 (1899), p. 47. Vallée Poussin, *Cours d'Analyse infinitésimale* (1903), p. 30. Schimmack, *Nova Acta*, Vol. XC, p. 5. Hamel, *Mathematische Annalen*, Vol. LX (1905), p. 459.

‡ *American Mathematical Monthly*, Vol. XVIII (1911), p. 198.

§ *Tidsskrift for Mathematik*, Vol. II, Ser. 4 (1878), p. 149.

‖ *Cours d'Analyse* (1821), p. 114.

¶ *American Mathematical Monthly*, Vol. XVI (1909), p. 180.

** *Transactions of the American Mathematical Society*, Vol. XVII (1916), p. 9.

A normal equation of order n is derived (in § 2) which is satisfied by every solution of any non-exceptional equation (1) of order n. This normal equation forms a foundation upon which the entire development of the theory of equation (1) is based. Any normal solution may be uniquely determined at each point of a dense set covering the complex plane if it is given at the vertices of a certain triangular network (§§ 3, 4). It is shown in § 5 that *the normal solution analytic in the neighborhood of the point zero of the complex plane is an arbitrary polynomial in x of degree n.* It is also shown that *the normal solution continuous in the neighborhood of the point zero of the complex plane is an arbitrary polynomial in u and v of degree n where u and v are real and $x=u+v\sqrt{-1}$.* The normal solution analytic along any line in the finite complex plane is also an arbitrary polynomial in x of degree n and the normal solution continuous along any line in the finite complex plane is an arbitrary polynomial in u of degree n if the line is not parallel to the axis of imaginaries and an arbitrary polynomial in v of degree n if the line is not parallel to the axis of reals. The analytic and continuous solutions of (1) are found (§ 6) from the normal solution. Examples are exhibited in § 6 which show that equations of type (1) may have non-trivial continuous solutions, but no non-trivial analytic solutions, while other examples show that equations of type (1) may have analytic solutions which are also the most general continuous solutions. A converse theorem is briefly considered in § 7. It is shown (in § 8) that *if a function $f(x)$ satisfying an equation of type* (1) *has a point of discontinuity in the finite complex plane* [*or on any line in the finite complex plane*] *it has a point of discontinuity in every region* [*interval*] *of the plane* [*line*], *however small.*

Equation (1) is employed (§ 9) to solve certain equations of the type

$$\sum_{i=1}^{n}\phi_i(x,y)f(\alpha_i x+\beta_i y)+\phi_{n+1}(x,y)f(x)+\phi_{n+2}(x,y)f(y)+\phi_{n+3}(x,y)=0,$$

where the ϕ's are known functions. Equation (1) is also employed (§ 10) to find all analytic solutions, and in some cases, the continuous solutions of binomial equations of the type

$$\prod_{i=1}^{k}[f(\alpha_i x+\beta_i y)]^{\gamma_i}=C\prod_{i=k+1}^{n+1}[f(\alpha_i x+\beta_i y)]^{\gamma_i}[f(y)]^{\gamma_{n+2}},$$

where no α is zero, C is a constant and the γ's are constants of which the real parts are positive. Pexider* used the first Cauchy equation to solve

$$f(x)+\phi(y)=\psi(x+y).$$

In § 11 the method of obtaining the solutions of (1) is used to solve the equation

$$\sum_{i=1}^{n}\gamma_i f_i(\alpha_i x+\beta_i y)+\gamma_{n+1}f_{n+1}(x)+\gamma_{n+2}f_{n+2}(y)=0,$$

* *Monatshefte für Mathematik und Physik*, Vol. XIV (1903), p. 293.

of which the equation considered by Pexider is a special case. It is proved that when no two arguments of f in the foregoing equation are proportional, each continuous solution f is a polynomial of degree not greater than n.

§ 2. *Reduction to a Normal Form.*

The solution $f(x)$ of the general n-th order equation

$$\sum_{i=1}^{n}\gamma_i f(\alpha_i x+\beta_i y)+\gamma_{n+1}f(x)+\gamma_{n+2}f(y)=0, \tag{1}$$

is contained in that of an equation of the same form and same order [equation (6) below] in which each α, β, γ is a given integer. The derivation of (6) from (1) is accomplished by elimination.

If (1) is subtracted from the equation derived from (1) by replacing y by $y+t_{n+1}$, the result is

$$\sum_{i=1}^{n}\gamma_i[f(\alpha_i x+\beta_i y+\beta_i t_{n+1})-f(\alpha_i x+\beta_i y)]+\gamma_{n+2}[f(y+t_{n+1})-f(y)]=0. \tag{2}$$

If (2) is subtracted from the equation derived from (2) by replacing x by $x-\beta_1 t_1$ and y by $y+\alpha_1 t_1$, the result is

$$\begin{aligned}\sum_{i=2}^{n}\gamma_i[f(\alpha_i x+\beta_i y+\Delta_{1i}t_1+\beta_i t_{n+1})-f(\alpha_i x+\beta_i y+\Delta_{1i}t_1)\\ -f(\alpha_i x+\beta_i y+\beta_i t_{n+1})+f(\alpha_i x+\beta_i y)]\\ +\gamma_{n+2}[f(y+\alpha_1 t_1+t_{n+1})-f(y+\alpha_1 t_1)-f(y+t_{n+1})+f(y)]=0,\end{aligned} \tag{3}$$

where

$$\Delta_{ji}=\alpha_j\beta_i-\alpha_i\beta_j.$$

It is easily seen that this is true because the given substitutions leave $\alpha_1 x+\beta_1 y$ unchanged, but replace $\alpha_i x+\beta_i y$, $i\neq 1$, by

$$\alpha_i x+\beta_i y+\alpha_1\beta_i t_1-\alpha_i\beta_1 t_1=\alpha_i x+\beta_i y+\Delta_{1i}t_1.$$

In general, if an equation (a) results after such eliminations, then each argument of f in (a) is a linear expression in x, y and certain t's, the subscripts of the t's corresponding to those in the terms eliminated. The substitution of

$$x-\beta_j t_j \text{ for } x \quad\text{and}\quad y+\alpha_j t_j \text{ for } y \tag{4}$$

gives rise to an equation (b) which differs from (a) by having each $\alpha_i x+\beta_i y$ of (a) replaced by $\alpha_i x+\beta_i y+\Delta_{ji}t_j$. Since $\Delta_{jj}=0$ and since (4) does not affect the t's that are found in (a), it follows that the difference formed by subtracting (a) from (b) contains no term for which i is equal to the fixed integer j. Since n is finite these eliminations may be continued until only terms having the coefficients $\pm\gamma_{n+2}$ remain. Moreover, the order of elimination of the terms for which $i=1, 2, \ldots, n$, is immaterial.

It is obvious that the equation resulting from these eliminations is linear and homogeneous. Since $\gamma_{n+2} \neq 0$ by the assumption that (1) is of order n, the equation may be simplified by dividing by γ_{n+2}. It is easily seen that there are $(n+1)!/k!(n+1-k)!$ distinct terms in which the coefficient of f is $(-1)^k$, and in which the argument of f is obtained by omitting k terms after the first from $y+\alpha_1 t_1+\alpha_2 t_2+\ldots\ldots+\alpha_n t_n+t_{n+1}$. Moreover, no other such terms are possible. This is true for $k=0, 1, \ldots\ldots, n+1$. The substitutions t_1 for $\alpha_1 t_1$, t_2 for $\alpha_2 t_2$, $\ldots\ldots$, t_n for $\alpha_n t_n$, t_{n+2} for y, serve to completely determine an equation independent of the original α's, β's and γ's. If Σ_k denotes the sum of the $(n+1)!/k!(n+1-k)!$ terms in which the arguments are formed by omitting k t's from $\Sigma_{i=1}^{n+1} t_i$, then the equation which is satisfied by $f(x)$ is

$$\Sigma_0-\Sigma_1+\Sigma_2-\ldots\ldots+(-1)^n\Sigma_n+(-1)^{n+1}\Sigma_{n+1}=0. \tag{5}$$

Equation (5) involves $n+2$ independent variables. In order to obtain a normalized equation having the same form and order as (1), let $t_1=t_2=\ldots\ldots=t_{n+1}=x$ and $t_{n+2}=y$, whence (5) becomes*

$$f[(n+1)x+y]+\ldots\ldots+(-1)^{n+1-k}\frac{(n+1)!}{k!(n+1-k)!}f(kx+y)$$
$$+\ldots\ldots+(-1)^n f(x+y)+(-1)^{n+1}f(y)=0. \tag{6}$$

From the foregoing considerations we see that *every solution $f(x)$ of equation* (1) *is a solution of equation* (5) *and of the normal equation* (6).

While the solutions $f(x)$ of (1) are included among those of (6), it is not necessarily true that the solutions of (6) are included in those of (1). The following example suffices to prove this statement. Equation (6) for $n=2$ is

$$f(3x+y)-3f(2x+y)+3f(x+y)-f(y)=0. \tag{7}$$

As will be shown in § 5, the most general continuous solution of (7) over the finite complex x-plane is

$$f(x)=au^2+buv+cv^2+du+ev+f, \tag{8}$$

where $x=u+v\sqrt{-1}$ (u and v real) and a, b, c, d, e and f are arbitrary constants. From the above considerations we see that the solution of any second order equation of form (1) is included among those of (8). However, substitution shows that for the equation

$$f(2x+y)-2f(x+y)-2f(x)+f(y)=0$$

---

* The results in (5) and (6) can be rendered more precise in special cases. If $\beta_j/a_j=\beta_k/a_k$, $\Delta_{jk}=0$ and the substitution of $x-\beta_j t_j$ for x and $y+a_j t_j$ for y leaves $a_k x+\beta_k y$, as well as $a_j x+\beta_j y$, unchanged. Hence the elimination of terms for which $i=j$ also eliminates all terms for which $i=h$ where h is any value for which $\Delta_{jh}=0$. But when $\Delta_{jh}=0$, $\beta_j/a_j=\beta_h/a_h$, and therefore this proportionality is a necessary and sufficient condition for the simultaneous elimination of terms for more than one value of i. Hence *the solution of any m-th order equation* (1) *in which no a is zero, is contained in the solution of equation* (5) *or of equation* (6) *where n is the number of distinct ratios β_i/a_i in* (1).

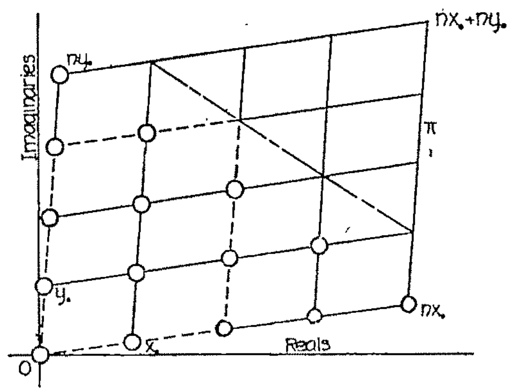
we have $d=e=f=0$. In fact, such an equation need have no other than the trivial solution $f(x)\equiv 0$ as is shown by the equation

$$f(2x+y)-f(x+y)-2f(x)+f(y)=0.$$

§ 3. *Determination of the Normal Solution at the Vertices of a Network.*

The function $f(x)$ to be considered in this section and in §§ 4, 5 is the solution of the normal equation developed in § 2. It is to be observed that if f is given for the arguments ky_0, x_0+ky_0, $2x_0+ky_0$, ..., nx_0+ky_0, then f is known for the argument $(n+1)x_0+ky_0$ by (6). By putting $y=jx_0+ky_0$ for successive values 1, 2, ..., of j, f is determined by (6) for any argument mx_0+ky_0, where m is any positive integer. Similarly, by successively giving to j the values -1, -2, ..., f is determined by (6) for any argument mx_0+ky_0, where m is a negative integer. By interchanging x and y in (6) it is easily seen that f is known for all arguments mx_0+ky_0, where k is any integer or zero, if it is given for the arguments mx_0, mx_0+y_0, mx_0+2y_0, ..., mx_0+ny_0. Hence, if f is given for the arguments mx_0+ky_0, m, $k=0, 1, \ldots, n$, then it is known by (6) for all arguments mx_0+ky_0, where m is any integer whatever and $k=0, 1, \ldots, n$, and finally, it is known by (6) for all arguments mx_0+ky_0, where m and k are any integers whatever.

It will now be proved that if f is known for the arguments mx_0+ky_0, where m and k are zero or positive integers and $m+k<n+1$, then it is also known for the arguments mx_0+ky_0, where m, $k=0, 1, \ldots, n$. The figure illustrates the case for which $n=4$. The points designated by the small circles



represent the arguments for which f is supposed known. The parallelogram formed by the lines joining the points 0, nx_0, nx_0+ny_0, and ny_0 will be denoted by π. The parallelogram π contains, on and within its boundary, all the points mx_0+ky_0, m, $k=0, 1, \ldots, n$, and no other points as vertices. It is required, therefore, to find f at the vertices of π not designated by small circles.

Equation (5) furnishes a simple means of solution of the present problem. Let $t_1=t_2=\ldots=t_q=x_0$, $t_{q+1}=\ldots=t_{n+1}=y_0$ and $t_{n+2}=0$. By these substitutions every argument in (5) is of the form mx_0+ky_0 where m and k are zero or positive integers and $m+k<n+1$, except in the first term in which $m+k=n+1$. Moreover, $m\leq q$ and $k\leq n+1-q$; hence f is determined for the argument $qx_0+(n+1-q)y_0$ in terms of linear combinations of its values for arguments represented by vertices within or on the boundary of the parallelogram bounded by lines joining the four points 0, qx_0, $qx_0+(n+1-q)y_0$, $(n+1-q)y_0$, (represented in the figure by dotted lines for $q=2$). By giving q the values 1, 2, ..., n, in succession, f is determined at each vertex of π which lies on the line joining the points x_0+ny_0 and nx_0+y_0 (the dot-and-dash line in the figure). It is to be noticed that each determination is made from a linear equation in one unknown with unit coefficient. Hence, the functional values so found are unique.

Having determined f for the arguments mx_0+ky_0 in π for which $m+k=n+1$, it is easy to determine f at the vertices for which $m+k=n+2$. In (5) let $t_1=t_2=\ldots=t_q=x_0$ and $t_{q+1}=\ldots=t_{n+2}=y_0$. Thus f is determined for the argument $qx_0+(n+2-q)y_0$ in terms of arguments represented by vertices within and on the boundary of the parallelogram formed by the four lines joining the points 0, qx_0, $qx_0+(n+2-q)y_0$, $(n+2-q)y_0$. This determination is also made by means of a linear equation involving but one unknown with the coefficient $+1$. Hence, no indetermination can be introduced. By giving q the successive values 2, 3, ..., n, f is determined for all arguments in π which lie on the line joining the points $2x_0+ny_0$ and nx_0+2y_0.

Proceeding in this manner f is determined for all the remaining arguments of π by giving t_{n+2} the values $2y_0$, $3y_0$, ..., $(n-1)y_0$ successively. For each value hy_0 of t_{n+2}, f must be determined for all arguments

$$qx_0+(n+h+1-q)y_0, \quad q=h+1, \quad h+2, \ldots, n,$$

before giving t_{n+2} the value $(h+1)y_0$. As before, each determination is made by a linear equation in one unknown with coefficient $+1$. Hence, f is uniquely determined at all the vertices of π, and we have the following result:

Every solution $f(x)$ of the normal equation (6) *is known for the points mx_0+ky_0 where m and k are any integers of zero if it is given at the points mx_0+ky_0 for which m and k are positive integers or zero and $m+k<n+1$.*

§ 4. *Determination of the Normal Solution at a Dense Set of Points.*

It will now be shown that if the solution of the normal equation is known at all points mx_0+ky_0, where m and k are integers, then it may be found for

$\frac{1}{2}mx_0+\frac{1}{2}ky_0$, and finally for $2^{-s}mx_0+2^{-s}ky_0$ for all integers m, k and s. If f is found for certain appropriate linear combinations of $2^{-s}x_0$ and $2^{-s}y_0$, then it is evident that $2^{-s}x_0$ and $2^{-s}y_0$ may be regarded as were x_0 and y_0 in § 3. Hence, if it is most convenient to determine f at $2^{-s}mx_0+2^{-s}ky_0$, $m, k=0, 1, \ldots, 2n$, it is also sufficient, in view of the argument of § 3, for f will then be known at these points for all integers m and k. Furthermore, if it is shown that f is known at $2^{-1}mx_0+2^{-1}ky_0$ for all integers m and k when it is known at mx_0+ky_0, then by regarding $2^{1-s}x_0$ and $2^{1-s}y_0$ as x_0 and y_0 for the successive values $s=2, 3, \ldots$, it is seen that f is known at $2^{-s}mx_0+2^{-s}ky_0$ for all integers m, k and s. Therefore, since the points $2^{-s}mx_0+2^{-s}ky_0$, m, k and s any integers whatever, form a dense set, it is sufficient to show that f is known at $\frac{1}{2}mx_0+\frac{1}{2}ky_0$, $m, k=0, 1, \ldots, 2n$, to show that f is known at all points of a dense set.

Supposing f known at $\frac{1}{2}mx_0+\frac{1}{2}ky_0$, $m, k=0, 2, \ldots, 2n$, it is only required to learn f at $\frac{1}{2}mx_0+\frac{1}{2}ky_0$, $m, k=1, 3, \ldots, 2n-1$. Replacing x by $\frac{1}{2}x_0$ and y by $\frac{1}{2}hx_0+ky_0$ in (6), n equations are obtained by giving h the values $n-1, \ldots, 1, 0$. The equations are linear in f. Alternate terms involve the arguments $\frac{1}{2}mx_0+ky_0$, $m=1, 3, \ldots, 2n-1$, for which f is unknown. The n equations may be considered as linear equations in n unknowns. It is easily seen that the determinant of the coefficients of the unknowns is $(-1)^{\nu}\Delta_n$, where ν is $n/2$ or $(n+1)/2$ according as n is even or odd, and Δ_n is a determinant of binomial coefficients such that the element in the i-th row and j-th column is

$$(n+1)!/(2j-i)!(n-2j+i+1)!,$$

unless $2j<i$ or $2j>n+i+1$, in which case it is zero. Subtracting the $(i+1)$-th row from the i-th row, i having the values $1, 2, \ldots, n-1$, in order, the element in the i-th row and j-th column, $i\neq n$, becomes

$$(n-4j+2i+2)\,(n+1)!/(2j-i)!(n-2j+i+2)!,$$

unless $2j<i$ or $2j>n+i+2$, in which case it is zero. Adding to the j-th column the sum of the preceding columns, giving j the values $n, n-1, \ldots, 2$ in order, Δ_n assumes a form in which the element in the i-th row and j-th column, $i\neq n$, is

$$S=\sum_{h=1}^{j}(n-4h+2i+2)\,(n+1)!/(2h-i)!(n-2h+i+2)!.$$

If for any given value of j this sum is $n!/(2j-i)!(n-2j+i)!$, unless $2j<i$ or $2j>n+i$, in which case it is zero, it is easily shown that for $j+1$ it is $n!/(2j-i+2)!(n-2j+i-2)!$, unless $2j<i-2$ or $2j>n+i-2$, in which case it is zero. For $2j-i=0$ or $2j-i=1$ the value of S is merely the first nonvanishing term, that is, 1 or n, respectively. Since $n!/(2j-i)!(n-2j+i)!$ is

equal to 1 for $2j-i=0$, and equal to n for $2j-i=1$, it follows by induction that S has the value

$$S=(n)!/(2j-i)!(n-2j+i)!,$$

unless $2j<i$ or $2j>n+i$, in which case it is zero. For the n-th column the value of S is always zero for $i\neq n$ since $2j>n+i$. For $i=n$, it is clear that the elements are affected only by the process of addition, and since every alternate term of $(1+1)^{n+1}$ is involved, the last term in the n-th column is 2^n. Hence, Δ_n has been so transformed that the principal $(n-1)$-rowed minor found by deleting the last row and last column of Δ_n is Δ_{n-1}, and the last column consists exclusively of zeros with the single exception of the element 2^n in the n-th row. Therefore $\Delta_n=2^n\Delta_{n-1}$. It is evident that $\Delta_2=2^{1+2}$ (Δ_1 is trivially 2). Therefore $\Delta_n=2^{1+2+\dots+n}=2^{\frac{1}{2}n(n+1)}$ which is distinct from zero for all finite values of n. Since $(-1)^r\Delta_n\neq 0$, it follows immediately that f is uniquely determined at the points $\frac{1}{2}mx_0+ky_0$, $m=1, 3, \dots, 2n-1$, and hence for $m=1, 2, \dots, 2n$.

Having determined f at the points $\frac{1}{2}mx_0+ky_0$, $m=1, 2, \dots, 2n$, it is only necessary to let $x=\frac{1}{2}y_0$ and $y=\frac{1}{2}hy_0+\frac{1}{2}mx_0$ in (6), letting h take the values $n-1, \dots, 1, 0$, to determine f at the points $\frac{1}{2}mx_0+\frac{1}{2}ky_0$, m, $k=0, 1, \dots, 2n$, for $(-1)^r\Delta_n$ is again the determinant of the coefficients of the unknown terms.

Combining the results of §§ 3 and 4, it may be stated that f *is determined at each point of the dense set* $2^{-s}mx_0+2^{-t}ky_0$ *if it is known at the points* mx_0+ky_0 *for which* m *and* k *are positive integers or zero and* $m+k<n+1$.

§ 5. *Solutions of the Normal Equation.*

Equation (6) may be written in the form

$$\sum_{j=0}^{n+1}(-1)^j\frac{(n+1)!}{j!(n-j+1)!}f[(ju+s)+\sqrt{-1}(jv+t)]=0,$$

where $x=u+v\sqrt{-1}$, $y=s+t\sqrt{-1}$, and u, s, v, t are real. Let us seek a solution $f(x)$ of this equation in the form

$$f(x)=\sum_{h=0}^{N}\sum_{q=0}^{h}c_{qh}u^qv^{h-q}, \tag{9}$$

in which the coefficients c_{qh} are constants. Putting this value of $f(x)$ in (6) we have

$$\sum_{j=0}^{n+1}(-1)^j\frac{(n+1)!}{j!(n-j+1)!}\sum_{h=0}^{N}\sum_{q=0}^{h}c_{qh}(ju+s)^q(jv+t)^{h-q}=0.$$

The terms involving $u^{q-p}s^pv^{h-q-r}t^r$ are

$$\sum_{j=0}^{n+1}\left[(-1)^j\frac{(n+1)!}{j!(n-j+1)!}j^{h-p-r}\right]c_{qh}\frac{q!}{p!(q-p)!}\frac{(h-q)!}{r!(h-q-r)!}u^{q-p}s^pv^{h-q-r}t^r.$$

Since no term outside of the brackets involves j, it is evident that the given expression vanishes, for non-zero c_{qh}, u, s, v and t, when and only when

$$B_{qh} = \sum_{j=0}^{n+1}\left[(-1)^j \frac{(n+1)!}{j!(n-j+1)!} j^{h-p-r}\right]$$

vanishes. Except for sign the quantity B_{qh} is the $(n+1)$-th difference of x^{h-p-r} for $x=0$. The degree of each difference is one less than that of the preceding difference. Therefore B_{qh} is zero when $h-p-r \leqq n$. The particular value h obtained from $h-p-r$ by putting $p=r=0$ must be included in the discussion of B_{qh}. Since N was taken as the largest value of h, it follows that $f(x)$, as given by (9), satisfies (6) if $N=n$. Moreover, the value of c_{qh} is arbitrary.

Equation (9) shows that there are $\Sigma_{h=0}^{n}(h+1) = \frac{1}{2}(n+1)(n+2)$ arbitrary constants c_{qh} which may be assigned at will. By §§ 3 and 4 it has been shown that when f is known at the $1+2+\ldots.+(n+1) = \frac{1}{2}(n+1)(n+2)$ points mx_0+ky_0, m and k positive integers or zero and $m+k<n+1$, then it is known over a dense set of points covering the entire finite plane provided x_0 and y_0 are not collinear with the point zero. Since $f(x)$, as given by (9), is continuous, it is only necessary to prove that each c_{qh} is uniquely determined by assigning f at the given points mx_0+ky_0 to know that $f(x)$ is the most general continuous solution of (6) over the finite complex plane. This can be done by direct substitution,* but inasmuch as the determinant so formed is unwieldy, it is more easily accomplished by observing that the properties sought for any desired oblique network are readily deduced by a projective transformation from similar properties of a square array, on the axes of reals and imaginaries with the units 1 and i, provided only that x_0 and y_0 do not lie on the same straight line through the zero-point. Confining attention to the rectangular array mentioned, write

$$f(x) = \sum_{h=0}^{n} \sum_{q=0}^{h} A_{qh} u^{(q)} v^{(h-q)}$$

where

$$u^{(q)} = u(u-1)\ldots.(u-q+1).$$

For any integral value of u less than q, $u^{(q)}=0$. Beginning with $x=0$ and proceeding outward through a triangular network similar to that employed in § 3, it is possible to determine an A_{qh} with each point of the net. Having determined the A_{qh}'s, it is only necessary to expand and collect the terms of the expression for $f(x)$ and compare with the expression involving the c_{qh}'s to completely determine each c_{qh}.

* The value of the determinant of the coefficients may be shown to be

$$(v_0 s_0 - u_0 t_0)^{\frac{1}{6}n(n+1)(n+2)} \prod_{a=0}^{n-1} [(n-a)!]^{2(a+1)}.$$

Thus we see that *the most general solution* $f(x) \equiv f(u+iv)$ *of equation* (6) *continuous over the finite complex x-plane is an arbitrary polynomial in u and v of degree n.*

The analytic solution of (6) over the finite complex plane is that special case of the general continuous solution for which

$$f(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots.$$

Replacing x by $u+iv$, it is seen at once that $a_k u^k$ must be zero for $k > n$, and hence $a_k = 0$, $k > n$. Moreover, $a_k = c_{k0}$, $k \leqq n$. Hence *the most general analytic solution of* (6) *is an arbitrary polynomial in x of degree n.**

To obtain the most general continuous solution of (6) along any line in the finite complex plane, it is only necessary to observe that by the argument of §§ 3 and 4 it was proved that f is known at a dense set of points on the line if it is known at $n+1$ points of the line which are separated by some convenient unit. The argument at the beginning of this section shows that along any line not parallel to the axis of imaginaries,

$$f(x) = \sum_{j=0}^{n} a_j u^j,$$

where each a is arbitrary, satisfies (6). Since $f(x)$ is continuous, it remains only to show that the a's are determined by the functional values at the $n+1$ points on the line to know that $f(x)$ is the most general continuous solution of (6) along the line. Substitution shows immediately that the a's are uniquely determined by the $n+1$ values of f on the line. Hence *the most general solution of* (6) *continuous along any line not parallel to the axis of imaginaries is an arbitrary polynomial in u of degree n.*

Similarly, *the most general solution of* (6) *continuous along any line not parallel to the axis of reals is an arbitrary polynomial in v of degree n.*

§ 6. *Solutions of the Original Equation.*

The determination of the existence of any solution $f(x)$ of the equation

$$\sum_{i=1}^{n} \gamma_i f(\alpha_i x + \beta_i y) + \gamma_{n+1} f(x) + \gamma_{n+2} f(y) = 0, \tag{1}$$

and the determination of $f(x)$ if it exists, is accomplished by substituting the corresponding solution of the normal equation (6) in (1). It has been shown in § 5 that the general solution of (6) analytic over the finite complex x-plane

---

* That the analytic solution $f(x)$, if it exists, is a polynomial of degree not greater than n, is readily seen by direct differentiation of (1). If differentiation is made with respect to each of the variables y, $a_j x + \beta_j y$, $j = 1, 2, \ldots, n$ it results that $f^{(k)}(y) = 0$ whenever $k > n$, because the arguments are independent in pairs which are not proportional.

or along any line in the finite complex x-plane is an arbitrary polynomial in x of degree n. Hence, the determination of the general solution of (1) analytic over the finite complex x-plane or along a line in the finite complex x-plane, if it exists, is accomplished by substituting

$$f(x)=c_0+c_1x+\ldots+c_nx^n$$

in (1). Since the result of this substitution is an identity, the coefficient of any power of the variables must vanish.

It is convenient to consider the terms of degree n independently of the remaining terms. These terms are given by

$$c_n[\sum_{i=1}^{n}\gamma_i(\alpha_ix+\beta_iy)^n+\gamma_{n+1}x^n+\gamma_{n+2}y^n]=0.$$

Now c_n is necessarily zero unless

$$[\sum_{i=1}^{n}\gamma_i(\alpha_ix+\beta_iy)^n+\gamma_{n+1}x^n+\gamma_{n+2}y^n]\equiv 0.$$

Placing the coefficients of this identity equal to zero, we have

$$\sum_{i=1}^{n}\alpha_i^n\gamma_i+\gamma_{n+1}=0,\quad \sum_{i=1}^{n}\alpha_i^{n-k}\beta_i^k\gamma_i=0,\quad \sum_{i=1}^{n}\beta_i^n\gamma_i+\gamma_{n+2}=0,\quad k=1,2,\ldots,n-1. \tag{10}$$

Since only the ratios of the γ's are significant, it implies no loss of generality to assume $\gamma_{n+2}=-1$. Under this assumption equations (10) may be employed to express the remaining γ's in terms of the α's and β's, provided $f(x)$ contains a non-vanishing term of degree n. If we write $r_i=\alpha_i/\beta_i$, then for $i<n+1$

$$\gamma_i=r_1r_2\ldots r_n/\beta_i^n r_i\prod_{h=1}^{n}{}'(r_h-r_i),$$

where the prime indicates that h does not take the value i. Solution also gives

$$\gamma_{n+1}=(-1)^n r_1r_2\ldots r_n.$$

Therefore, if c_nx^n, $c_n\neq 0$, *is a term of the analytic solution* $f(x)$ *of* (1), *it is necessary and sufficient that* (1) *may be written in the form*

$$\sum_{i=1}^{n}\frac{r_1r_2\ldots r_n}{\beta_i^n r_i\prod_{h=1}^{n}{}'(r_h-r_i)}f(\alpha_ix+\beta_iy)+(-1)^n r_1r_2\ldots r_nf(x)-f(y)=0,$$

where $r_i=\alpha_i/\beta_i$.

If the solution of (1) includes the term c_mx^m, $m<n$ and $c_m\neq 0$, then

$$\sum_{i=1}^{n}\alpha_i^m\gamma_i+\gamma_{n+1}=0,\quad \sum_{i=1}^{n}\alpha_i^{m-k}\beta_i^k\gamma_i=0,\quad \sum_{i=1}^{n}\beta_i^m\gamma_i+\gamma_{n+2}=0,\qquad k=1,2,\ldots,m-1.$$

Since there are but $m+1$ linear equations in the γ's, they may be used to

express $m+1$ γ's in terms of the α's, β's and remaining γ's. For the first $m+1$ γ's in terms of the remaining quantities, solution gives

$$\gamma_i=\frac{-\sum_{j=m+2}^{n}\beta_j^m\gamma_j\prod_{h=1}^{m+1}{}'(r_j-r_h)-\gamma_{n+1}+(-1)^{m+1}r_1r_2\ldots r_{i-1}r_{i+1}\ldots r_{m+1}\gamma_{n+2}}{\beta_i^m\prod_{h=1}^{m+1}{}'(r_i-r_h)}, \tag{11}$$

where the prime indicates that h does not take the value i. Hence, *if* c_mx^m, $m<n$ *and* $c_m\neq 0$, *is a term of the analytic solution* $f(x)$ *of* (1), *it is necessary and sufficient that each of the first* $m+1$ γ's *has the value given in* (11).

The computation involved in finding the most general solution of (1) continuous over the entire finite plane is so tedious as to make it expedient to give results only for the general second order equation. The problem for any equation is merely a matter of substitution and algebraic computation. For the second order equation

$$\gamma_1f[(a_{11}+a_{12}i)x+(b_{11}+b_{12}i)y]$$
$$+\gamma_2f[(a_{21}+a_{22}i)x+(b_{21}+b_{22}i)y]+\gamma_3f(x)+\gamma_4f(y)=0,$$

where $a_{11}, a_{12}, b_{11}, b_{12}, a_{21}, a_{22}, b_{21}, b_{22}$ are real and $i=\sqrt{-1}$, the normal solution with which substitution must be made is

$$c_{00}+c_{10}u+c_{01}v+c_{20}u^2+c_{11}uv+c_{02}v^2.$$

This substitution shows that c_{00} may be assigned different from zero when and and only when $\gamma_1+\gamma_2+\gamma_3+\gamma_4=0$. It also shows that if we write

$$A_1=\gamma_1a_{11}+\gamma_2a_{21}+\gamma_3,\quad B_1=\gamma_1a_{12}+\gamma_2a_{22},\quad C_1=\gamma_1b_{11}+\gamma_2b_{21}+\gamma_4,\quad D_1=\gamma_1b_{12}+\gamma_2b_{22},$$

then for c_{10} and c_{01} to be independently arbitrary $A_1=B_1=C_1=D_1=0$. However, $c_{10}=c_{01}k_1$ if $A_1=B_1k_1$, $C_1=D_1k_1$, B_1 or $D_1\neq 0$, and $k_1^2=-1$. Furthermore, if we write,

$$\begin{array}{lll} A=\gamma_1a_{11}^2+\gamma_2a_{21}^2+\gamma_3, & B=\gamma_1a_{11}a_{12}+\gamma_2a_{21}a_{22}, & C=\gamma_1a_{12}^2+\gamma_2a_{22}^2, \\ D=\gamma_1b_{11}^2+\gamma_2b_{21}^2+\gamma_4, & E=\gamma_1b_{11}b_{12}+\gamma_2b_{21}b_{22}, & F=\gamma_1b_{12}^2+\gamma_2b_{22}^2, \\ G=\gamma_1a_{11}b_{12}+\gamma_2a_{21}b_{22}, & H=\gamma_1a_{11}b_{11}+\gamma_2a_{21}b_{21}, & J=\gamma_1a_{12}b_{12}+\gamma_2a_{22}b_{22}, \\ & K=\gamma_1a_{12}b_{11}+\gamma_2a_{22}b_{21}, & \end{array}$$

$c_{20}=kc_{02}$ and $c_{11}=mc_{02}$,* then it is easy to show that

(i) $k=-1$ and $m=\pm 2i$ if
 (a) $A+C$, $D+F$, $K-G$, or $H+J\neq 0$,
 (b) B, E, or $J-H\neq 0$,
 (c) $A-C=Bm$, $D-F=Em$, and $2(K+G)=(J-H)m$.

* The results desired here allow c_{02} to be different from zero, and no attempt is made to discuss the possibilities in case $c_{02}=0$.

(ii) $k=-1$ and $m=0$ if
  (a) A, D, H, or $G \neq 0$,
  (b) B or $E \neq 0$,
  (c) $C=A$, $F=D$, $J=H$, and $G+K=0$.

(iii) $k=-1$ and m is arbitrary if
  (a) A, D, H, or $G \neq 0$,
  (b) $B=E=0$, $C=A$, $F=D$, $J=H$, and $G+K=0$.

(iv) $k=1$ and $m=0$ if
  (a) A^2+B^2, D^2+E^2, or $G^2+H^2 \neq 0$,
  (b) $C=-A$, $F=-D$, $K=G$, and $J=-H$.

The following equations, constructed on the basis of this information, answer interesting questions. The equation

$$3f[(1+2i)x+(3+i)y]-f[(1+3i)x+(6+3i)y]-5f(x)+15f(y)=0$$

has the general continuous solution $f(x)=c_{02}(u^2+v^2)$ showing that *an equation may have a continuous solution, but no non-trivial analytic solution.*

The general solution of

$$(1+i)f[3x+(1-i)y]-3f[x+2y]-(6+9i)f(x)+(10+2i)f(y)=0,$$

continuous over the finite complex x-plane is $c_{20}(u^2+2iuv-v^2)=c_{20}x^2$, showing that *an equation may have its analytic solution as the most general continuous solution.*

The elimination outlined in the footnote of § 2 shows that every solution of an equation of form (1) having no α equal to zero is included in the corresponding solution of the normal equation whose order is equal to the number of distinct ratios β_i/α_i, $i=1, 2, \ldots, n$, in the equation of form (1). One case remains, namely, that in which each ratio β_i/α_i, α_i and β_i different from zero, is equal to at least one other such ratio and at least one α and one β, of different subscripts, are zero. That there are equations of this exceptional type which have infinite series solutions is proved by the equation

$$f(x+y)-f(ix+iy)-f(-ix)-f(-iy)+f(x)+f(y)=0, \quad i=\sqrt{-1},$$

which is satisfied by a series in positive integral powers of x^4 having arbitrary coefficients.

§ 7. *The Converse Theorem.*

It was shown in § 5 that any polynomial in x of degree m_j satisfies the normal equation of order m_j. It will now be proved that any polynomial $p(x)$ in x, of degree m_j, satisfies an equation (1) whose order n is not greater than the number of non-vanishing terms of $p(x)$, plus the sum of the degrees of

such terms, and whose α's, β's and γ_{n+1} and γ_{n+2} may be assigned at will, provided a certain determinant Δ of the α's and β's is not zero as a consequence.

Suppose that

$$p(x) = a_1 x^{m_1} + a_2 x^{m_2} + \ldots + a_j x^{m_j}.$$

The substitution of $p(x)$ for $f(x)$ in (1) gives an identity from which

$$\sum_{i=1}^{n} \alpha_i^{m_k} \gamma_i = -\gamma_{n+1}, \quad \sum_{i=1}^{n} \alpha_i^{m_k-k} \beta_i^k \gamma_i = 0, \quad \sum_{i=1}^{n} \beta_i^{m_k} \gamma_i = -\gamma_{n+2} \qquad k=1, 2, \ldots, m-1, \quad (12)$$

for the values $h=1, 2, \ldots, j$. The total number of independent equations is not greater than $j + \Sigma_{i=1}^{j} m_i$. Setting n equal to the number of independent equations, we have a system of non-homogeneous linear equations in n unknowns $\gamma_1, \gamma_2, \ldots, \gamma_n$, provided γ_{n+1} and γ_{n+2} are not both assigned equal to zero. A necessary and sufficient condition that the unknown γ's are uniquely determined in terms of the α's, β's and two assigned γ's is that the determinant Δ of the coefficients be different from zero. It is obvious that Δ is a polynomial in the α's and β's. Furthermore, it is at once evident that the term formed by the product of the elements in the principal diagonal is unique. Hence the polynomial in the α's and β's does not vanish identically. Therefore the α's and β's may be assigned in any way such that the polynomial Δ has a value different from zero.

If γ_{n+1} and γ_{n+2} are both assigned equal to zero, equations (12) form a system of n linear homogeneous equations in n unknowns, and the non-vanishing of Δ is a necessary and sufficient condition that each of the unknown γ's is zero. In this case the equation is trivially satisfied by $p(x)$. In general, then, *any polynomial $p(x)$ satisfies an infinity of equations* (1) *whose orders do not exceed the number of non-vanishing terms of $p(x)$ plus the sum of the degrees of such terms and whose α's, β's, γ_{n+1}'s and γ_{n+2}'s may be assigned at will, provided only that a polynomial Δ of the α's and β's does not vanish for the assigned values.*

§ 8. *Discontinuous Solutions.*

Any solution $f(x)$ of an equation (1) satisfies the normal equation whose order n is determined by (1). Suppose that the domain of $f(x)$ is any line in the finite complex x-plane, and that $f(x)$ is continuous in some interval of length $\delta > 0$ of the line. It is readily seen from the normal equation satisfied by f that if the first $n+1$ arguments are so chosen that they represent points in the interval while the remaining argument represents a point outside the interval, $f(x)$ is determined at the last-named point as the sum of continuous functions. Therefore f is continuous at the outside point. In this way f may be shown to be continuous at all points in the two intervals of length

δ/n which lie at the ends of the given interval. Therefore f is continuous in an interval of length $[(n+2)\delta]/n$. Any finite interval of length σ may be reached in this manner by a finite number of extensions of the interval of length δ. We may therefore state that *if $f(x)$ has a finite point of discontinuity on any line in the complex x-plane, it has a point of discontinuity in every interval of the line, however small.*

Suppose $f(x)$ is continuous in a region of the finite complex x-plane. A circle may be inscribed in the region such that $f(x)$ is continuous in the closed region of which the circle is the boundary. Suppose the radius of this circle is δ and consider a concentric circle of radius $[(n+1)\delta]/n$. By means of the normal equation f may be determined at any point in the area between the circles as the sum of $n+1$ continuous functions, namely, f at $n+1$ points in the circle of radius δ. Therefore f at the point between the circles is continuous. This is true for every point of the area between the circles, and hence f is continuous in the circle of radius $[(n+1)\delta]/n$. This process may obviously be repeated to prove that f is continuous in any finite region of the plane. Hence, *if $f(x)$ has a point of discontinuity in the finite complex x-plane, it has a point of discontinuity in every finite region of the plane.*

G. Hamel (*loc. cit.*) has exhibited a discontinuous solution* $f(x)$ of the Cauchy equation

$$f(x+y)=f(x)+f(y).$$

From the treatment in § 2 it is clear that f also satisfies

$$f(2x+y)-2f(x+y)+f(y)=0.$$

Replacing y by $hx+y$, multiplying the equation by $(-1)^h(n-1)!/h!(n-h-1)!$ for successive values $h=0,1,\ldots,n-1$, and adding the n equations so formed, it is easily seen that

$$\sum_{k=0}^{n+1}(-1)^k\frac{(n+1)!}{k!(n+1-k)!}f(kx+y)=0. \tag{6}$$

We therefore have a discontinuous solution of the normal equation for each order n.

§ 9. *Certain Types of Equations having Variable Coefficients.*

The functional equations that have been discussed may be employed to solve certain equations of the form,

$$\sum_{i=1}^{n}\phi_i(x,y)f(\alpha_i x+\beta_i y)+\phi_{n+1}(x,y)f(x)+\phi_{n+2}(x,y)f(y)+\phi_{n+3}(x,y)=0, \tag{13}$$

where the ϕ's are known functions. A general statement and a few examples suffice to indicate some of the equations that may be solved. Suppose there are k transformations

$$x=\gamma_j x'+\delta_j y',\quad y=\lambda_j x'+\mu_j y',$$

* This solution is obtained on the assumption of the validity of the Zermelo axiom.

which may be applied to (13) to obtain new equations such that if each equation is multiplied by a non-zero constant, the sum of them is of form (1). A solution of (13), if it exists, is included in the corresponding solution of the auxiliary equation of form (1) deduced from (13). In order to find a solution of (13) it is sufficient to substitute the solution of the auxiliary equation and compute the coefficients of the variables.

Equation (13) includes the non-homogeneous equation in which $\phi_i(x, y)$, $i=1, 2, \ldots, n+2$, is further restricted to be a constant. In this case it is obvious from equation (13) that $\phi_{n+3}(x, y)$ is a polynomial in x and y if an analytic solution exists, and a polynomial in u, v, s and t if a continuous solution exists. Such a non-homogeneous equation is*

$$f(x+y)=f(x)+f(y)+2xy. \tag{14}$$

Transformations which may be used to solve this equation are

$$x=x', \ y=x'-y' \quad \text{and} \quad x=x', \ y=y'-x'$$

whence, after dropping the primes,

$$f(2x-y)-f(x-y)-f(-x+y)-2f(x)+f(y)=0.$$

The arguments $x-y$ and $-x+y$ are proportional and the normal equation is therefore of order 2. Hence the general solution of (14), analytic over the finite complex x-plane is readily seen to be $f(x)=a_1x+x^2$, where a_1 is arbitrary. The general solution of (14) continuous over the finite complex x-plane is $f(x)=a_{10}u+a_{01}v+x^2$, where a_{10} and a_{01} are arbitrary.

Suppose all the ϕ's are constant and $\phi_{n+3}\neq 0$. If $\Sigma_{i=1}^{n+2}\phi_i\neq 0$, then $f(0)$ is finite and uniquely determined, and the transformation $f(x)=g(x)+f(0)$ may be employed to obtain an equation (1) of order n in $g(x)$. Hence $f(x)$ is a polynomial (in x, in u and v, in u or in v, as the case may be) of degree not greater than n. The transformation used in this case has the advantage of furnishing an auxiliary equation (1) whose order is not greater than that of the original equation (13).

An equation which illustrates reduction by interchanging arguments of f is

$$(\cos^2 x)f(x+y)+(\sin^2 x)f(x-y)-f(x)-f(y)-2(\cos 2x)xy=0.$$

If y is replaced by $-y$ the equation becomes

$$(\sin^2 x)f(x+y)+(\cos^2 x)f(x-y)-f(x)-f(-y)+2(\cos 2x)xy=0.$$

The sum of these equations is of form (1). The solution analytic over the finite complex x-plane of the equation having variable coefficients is $f(x)=x^2$. It is easily seen that this is the most general continuous solution.

The equation

$$2f(2x+y)+(x+y)f(x-y)+3f(x)-3f(y)=0$$

---

* *American Mathematical Monthly*, Vol. XXIV (1917), p. 178.

may be reduced by replacing x by $2x+y$ and y by $x+2y$, whence

$$2f(5x+4y)+3(x+y)f(x-y)+3f(2x+y)-3f(x+2y)=0,$$

and subtracting 3 times the original equation from the transformed equation. The reduced equation is

$$2f(5x+4y)-3f(2x+y)-3f(x+2y)-9f(x)+9f(y)=0.$$

It is readily seen that the equation with variable coefficient has no continuous solution.

If one of the first $n+2\phi$'s of (13) is variable while the remaining ϕ's are constant the product of that ϕ and the corresponding $f(\alpha_i x+\beta_i y)$ is a polynomial. The variable ϕ is therefore a rational function whose denominator is a factor of $f(\alpha_i x+\beta_i y)$. An equation illustrating this point is

$$2\left[\frac{x+y}{x-y}\right]^2 f(x-y)-f(x+2y)-f(x)+2f(y)=0.$$

The equation obtained by replacing x by $2x+y$ and y by $x+2y$ is

$$18\left[\frac{x+y}{x-y}\right]^2 f(x-y)-f(4x+5y)-f(2x+y)+2f(x+2y)=0.$$

The equation obtained by subtracting the transformed equation from 9 times the original one gives an equation (1) of order 3. The analytic solution $f(x)$ of the equation with a variable coefficient is then easily seen to be ax^2, where a is arbitrary.

§ 10. *Application to Binomial Equations.*

An application of linear functional equations having constant coefficients may also be made to certain equations of the form,

$$\prod_{i=1}^{k}[f(\alpha_i x+\beta_i y)]^{\gamma_i}=C\prod_{i=k+1}^{n+1}[f(\alpha_i x+\beta_i y)]^{\gamma_i}[f(y)]^{\gamma_{n+2}}, \qquad (15)$$

where C is a constant, the real part of each γ is positive and no α is zero. Let us first consider the solution $f(x)$ of (15) analytic at all points in the finite complex plane. Suppose that $f(x)$ has a zero at some point $x=a$. Let $y=a$ in (15). Then

$$\prod_{i=1}^{k}[f(\alpha_i x+\beta_i a)]^{\gamma_i}=0$$

for all values of x. Hence there is a finite region in which $f(x)$ has an infinity of zeros. But this is impossible since $f(x)$ is analytic throughout the finite plane. Therefore, *when $f(x)$ is analytic throughout the finite complex plane, and not identically zero, it is never zero.* The case of a continuous solution $f(x)$ of (15) presents more difficulty. It is evident that *if one member of* (15) *has either no factor or only one factor involving f, then $f(x)$ is never zero unless it is identically so.*

The function $\phi(x)=\log f(x)$, where it is understood that the principal determination of the logarithm is employed, is analytic or continuous with f when the latter has no zeros. Therefore, in each of the cases considered above, $\phi(x)$ satisfies the equation

$$\sum_{i=1}^{k}\gamma_i\phi(\alpha_i x+\beta_i y)-\sum_{i=k+1}^{n+1}\gamma_i\phi(\alpha_i x+\beta_i y)-\gamma_{n+2}\phi(y)-K+2s\pi i=0,$$

where K is the principal determination of $\log C$ and s is any integer. For any given s $\phi(x)$ is a polynomial of degree not greater than n. Furthermore, the argumentation of § 9 shows that a variation in s affects only the constant term of $\phi(x)$. Therefore, *in each of the cases considered above, $f(x)$ is an exponential function of the form*

$$f(x)=e^{P+k(s)}, \tag{16}$$

where P is a polynomial of degree not greater than n, and $k(s)$ is a constant depending on s. Thus we see that in general the solutions of (15) are given by (16) for the various possible values of $k(s)$.

The same result may be stated for the continuous solution $f(x)$ of (15), *when each member involves at least two factors containing f, provided $f(0)\neq 0$.* For $\phi(x)$ as defined, is continuous in some region about the point $x=0$ since $f(x)$ is different from zero in some such region. For a given s, therefore, $\phi(x)$ must be everywhere continuous in the finite complex plane because it satisfies a non-exceptional equation of type (1).

The equation,

$$\psi(x+y)\psi(x-y)=[\psi(x)\psi(y)]^2,$$

mentioned in § 1, is included in the last case considered. For suppose there is at least one point $x=b$ at which $\psi(x)$ is not zero. Let $x=y=b$. Then

$$\psi(2b)\psi(0)=[\psi(b)]^4\neq 0,$$

and $\psi(0)\neq 0$. It is easily seen that the solutions $\psi(x)$ analytic over the finite complex plane are

$$\psi(x)=e^{ax^2-s\pi i}=\pm e^{ax^2},$$

and the solutions $\psi(x)$ continuous over the finite complex plane are

$$\psi(x)=e^{c_{20}u^2+c_{11}uv+c_{02}v^2-s\pi i}=\pm e^{c_{20}u^2+c_{11}uv+c_{02}v^2},$$

where a and the c's are arbitrary constants.

§ 11. *Equations Involving More than One Function.*

Consider the equation

$$\sum_{i=1}^{n}\gamma_i f_i(\alpha_i x+\beta_i y)+\gamma_{n+1}f_{n+1}(x)+\gamma_{n+2}f_{n+2}(y)=0, \tag{17}$$

where no γ is zero and the f's are unknown, continuous, single-valued functions to be determined if possible so that (17) shall be identically satisfied by them. The functions f_i may or may not all be distinct. The method of elimination employed in § 2 is applicable to (17). If no α is zero it is evident then that f_{n+2} satisfies the normal equation of order n. Therefore f_{n+2} is a polynomial of degree not greater than n. Under the assumption that no α_i and no β_i are zero, and no two of the ratios β_i/α_i are equal, any term may be given the argument y by a linear transformation which makes no α and no β zero. In this case, therefore, *every function f_i of (17) is a polynomial of degree not greater than n.* To find any necessary restrictions on the coefficients of these polynomials, it is sufficient to substitute the n-th degree polynomials having general coefficients in (17), and to equate to zero the resulting coefficients of the variables. It is obvious that equation (1) is a special case of (17).

If some of the ratios β_i/α_i are equal it may be assumed without loss of generality that equation (17) is so arranged that functions having arguments of a common ratio are placed consecutively. If all the functions having arguments of a common ratio have subscripts i, such that $g \overline{\gtrless} i \overline{\gtrless} h$, then we may write

$$F_h(\alpha_h x+\beta_h y)=\sum_{i=g}^{h}\gamma_i f_i(\alpha_i x+\beta_i y).$$

Equation (17) may now be written

$$\Sigma F_h(\alpha_h x+\beta_h y)+F_{n+1}(x)+F_{n+2}(y)=0, \tag{18}$$

where no α and no β are zero, and no two ratios β_h/α_h are equal. We denote by $q+2$ the number of terms in the first member of (18). Each F is a polynomial of degree not greater than q. Each F therefore determines a non-homogeneous equation in certain f's having arguments differing by constant factors. If the f's of any F are identical, that is, if the f's of F are the same function, F determines a non-homogeneous mixed q-difference equation satisfied by f. The equations of type (1), which have no α equal to zero, but have some ratios β_i/α_i equal, or some β's zero, are special cases of (18) given by $F_{n+2}(y)=\gamma_{n+2}f(y)$. The equations of the exceptional case noted in the last paragraph of § 6 are equations of form (18) which have no F a constant multiple of a single f. In this connection it is interesting to note that the function f of the example in the paragraph cited satisfies the equations

$$f(x)-f(-ix)=f(x)-f(ix)=0,$$

whence

$$f(-ix)=f(ix)=f(-x)=f(x).$$

*Contributions to the Study of Oscillation Properties of the Solutions of Linear Differential Equations of the Second Order.**

BY R. G. D. RICHARDSON.

Introduction.

The study of boundary problems for linear differential equations of the second order dates back to the time of Euler and D'Alembert, with whom it arose in connection with problems of mathematical physics. Beginning with the fundamental paper of Sturm in 1836, there have been extensive investigations † in this field in recent years, notably by Klein, Bôcher, Stekeloff, Kneser, Hilbert and Birkhoff. Since the differential equation of the second order is of such fundamental importance in so many fields, and since similar general problems for equations of higher order can not be handled by processes so far devised, the invention of new methods and further investigation of the nature of solutions find ready justification.

The equation to be studied will be taken in the form

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+G(x,\lambda)y(x)\equiv(py_x)_x+G(x,\lambda)y=0, \qquad 0\leqq x\leqq 1, \qquad (1)$$

where $G(x, \lambda)$ is a function depending on a parameter λ. The solution $y(x)$ of this self-adjoint equation shall be subject to the self-adjoint boundary conditions

$$\left.\begin{aligned} &\alpha_1 y(0)+\alpha_2 y_x(0)+\alpha_3 y(1)+\alpha_4 y_x(1)=0,\\ &\beta_1 y(0)+\beta_2 y_x(0)+\beta_3 y(1)+\beta_4 y_x(1)=0,\\ &p(1)(\alpha_1\beta_2-\alpha_2\beta_1)=p(0)(\alpha_3\beta_4-\alpha_4\beta_3), \end{aligned}\right\} \qquad (2)$$

where the two sets of real coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ are linearly independent. The most important special cases of these boundary conditions are given by

$$y(0)=y(1)=0; \quad y(0)=y_x(1)=0; \quad y_x(0)=y(1)=0; \quad y_x(0)=y_x(1)=0. \qquad (2')$$

* Read before the American Mathematical Society, September 4, 1917.

† For existing methods and literature of the subject see Bôcher, *Encyklopädie Mathematischen Wissenschaften*, II A7a, *Proceedings International Congress of Mathematicians*, Vol. I (1912), p. 163, and "Leçons sur les Methodes de Sturm" (1917); Lichtenstein, *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXVIII, p. 113.

There are many interesting questions in regard to this linear problem. Do there exist parameter values λ such that there are solutions of (1) satisfying relations (2)? If so, how many are there, and how are they distributed? What is the nature of the corresponding solutions? How do the solutions vary with change of the coefficients of the equation and of the boundary conditions? When does the totality of solutions form a fundamental set in terms of which functions may be expanded?

Among the methods which have been used in studying the problem are: (1) Differential equations including the use of comparison, approximation and asymptotic expressions; (2) the minimum principle in the calculus of variations; (3) integral equations; (4) the theory of linear algebraic equations in an infinite number of variables;* (5) a limiting process with linear algebraic approximating difference equations (cf. § 1, IV). The methods developed in this memoir would fall under (1) and (2) and may be characterized as a free use of the differentiation of fundamental formulae with regard to the parameters involved in the equations and in the boundary conditions, together with the proof that under the conditions imposed certain integrals are positive.

Exact oscillation theorems for solutions under the boundary conditions (2) have been developed by Birkhoff † for the case of the special equation

$$y_{xx}+G(x,\lambda)y=0,\quad \frac{\partial G}{\partial\lambda}>0,\quad \lim_{\lambda=-\infty} G=-\infty,\quad \lim_{\lambda=+\infty} G=+\infty.$$

Another important special case of equation (1) in which $G(x,\lambda)$ contains the parameter linearly

$$(py_x)_x+(q+\lambda k)y=0, \tag{3}$$

has been studied very extensively. When $k(x)>0$ this equation may be reduced to a form included in that investigated by Birkhoff.

The *definite* case of (3), viz., when one of the integrals

$$\int_0^1 ky^2dx,\quad \int_0^1 (py_x^2-qy^2)\,dx$$

has one sign for all functions $y(x)$ considered, has been discussed in many phases by mathematicians since the time of Sturm. By means of his theory of integral equations Hilbert ‡ established the *existence* of characteristic

* Lichtenstein, *loc. cit.*

† *Transactions of the American Mathematical Society*, Vol. X (1909), p. 259. It should be remarked that both in the second and third lines from the end of the statement of the principal theorem (p. 260) instead of $p+1$ we should read $p-1$.

‡ "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen" (Teubner, 1912).

parameter values and characteristic solutions of equation (3) for this definite case. As will be shown in § 5 the boundary conditions which he used are normal forms of (2). Oscillation theorems for equation (3), under some of the simple boundary conditions (2′), have been established by various means; among others by setting up the corresponding calculus of variations problem, and interpreting the Jacobi criterion.*

When k has both signs and q is positive in at least a part of the interval and sufficiently large there, it is not necessary that either of the integrals in question be definite. This case, which we shall call the *non-definite*, was first discussed incidentally by the author in a paper† in which he was treating the problem of oscillation theorems for two equations with two parameters. He showed that when the boundary conditions are $y(0)=y(1)=0$, there exists an integer n_1 such that for $n<n_1$ there are *no real solutions* which have n zeros, while for $n>n_1$ there are *at least two.*‡ At that time all the principal results of §§ 2–4 were obtained, but were not published.

For the special equation (3) and $k>0$ the theorems proved by Birkhoff were rediscovered by Haupt§ in his dissertation. The oscillation theorem stated in this dissertation for the case that k changes sign is corrected in a later article,‖ and by using the methods which I had developed, various oscillation theorems for the general equation (1) are derived. It is also shown by means of expansion theorems that if the non-definite equation (3) be taken in a certain normal form there exists an integer n_2 such that for $n>n_2$ there are precisely two solutions with n zeros and satisfying the boundary conditions (2).

The object of the present memoir is to investigate the conditions to be imposed on $G(x,\lambda)$ regarded as a function of λ, so that definite oscillation theorems for solution of (1) may be determined. In § 2 an attempt is made to bring as close together as possible necessary conditions and sufficient conditions for a limited or unlimited number of oscillations. A criterion for the behavior of the zeros with change of the parameter is obtained in § 3. This permits the development of very general theorems for the unique existence of solutions vanishing at the end points and possessing a prescribed number of zeros, and also for the existence of two, and only two solutions of this nature.

---

* *Mathematische Annalen*, Vol. LXVIII (1910), p. 279.

† *Transactions of the American Mathematical Society*, Vol. XIII (1912), p. 22.

‡ That there were *exactly* two when $n>n_1$ was stated by the author in a paper in the *Mathematische Annalen*, Vol. LXXIII, p. 289, in which he was discussing oscillation theorems for three linear equations with three parameters. The subsequent results of the memoir were not affected by this error which was corrected by a note in Vol. LXXIV (1913), p. 312, of the same journal.

§ "Untersuchungen über Oszillationstheoreme" (Teubner, 1911).

‖ Haupt, *Mathematische Annalen*, Vol. LXXVI, p. 67.

These theorems contain as special cases all known results in this field, and some new special cases are set forth in detail.

The non-definite case of (3) and (1) is discussed in § 4. The question of whether there may be for a given oscillation number more than two parameter values for which there are solutions of (3) (2) is settled by giving an example in which for any n in an interval n_1, n_2 there are four values of λ corresponding. That there exists an integer n_3 such that for $n > n_2$, there are exactly two solutions is proved by a method entirely different from that of Haupt, and in some particulars it would seem that the resulting theorem is less satisfactory, in others more satisfactory than his. The theory is also extended to cover some corresponding cases of (1). Concerning the complex solutions which correspond to values of $n < n_1$ some theorems are derived.

In the latter half of the memoir a method is developed for obtaining the facts in regard to the solutions of the *general* equation (1) under the general boundary conditions (2). With this end in view § 5 is devoted to a reduction of the boundary conditions to normal forms by means of the usual transformation of the dependent variable y, which leaves the number of zeros unchanged. Each of these three normal forms,

$$\begin{array}{lll} \text{I.} & \sigma y(0)+y_x(0)=0, & \tau y(1)+y_x(1)=0; \\ \text{II.} & y(0)=hy(1), & hp(0)y_x(0)=p(1)y_x(1); \\ \text{III.} & y(0)=lp(1)y_x(0), & lp(0)y_x(0)=y(1) \end{array}$$

contains one or two parameters σ, τ, h, l. When special values 0 or ∞ are assigned to these parameters, the forms reduce to the simple cases (2′) for which the facts are readily obtainable from the developments of the earlier sections. The first form is of essentially different character from the others. In the latter the parameter h or l is a double-valued function of λ, which in general is real in sub-intervals only, while in the former each of the parameters σ, τ is a single-valued function of the other and of λ, and is real throughout. By letting λ vary, and calculating the rates of change of these parameters and of $G(x, \lambda)$, theorems of oscillation are derived (§§ 6–8) for boundary conditions in each of the normal forms. While detailed results are not given in all cases, these are immediate developments of the fundamental facts ascertained.

Any linear differential equation of the second order

$$\phi y_{xx}+\psi y_x+\theta y=0 \quad \text{or} \quad \phi y_{xx}+\psi y_x+(\lambda\theta_1+\theta_2)y=0, \quad \phi>0,$$

may be thrown into the corresponding self-adjoint form (1) or (3) on multiplying by the function $p(x)=e^{\int_0^x \frac{\psi}{\phi}dx}$; the corresponding self-adjoint boundary

conditions (2) are changed in form only by the substitution $p(0)=1$, $p(1)=e^{\int_0^1 \frac{\psi}{\phi}dx}$.

The oscillation theorems remain unchanged if the variables are subjected to the usual transformations of the dependent and independent variables $y=\eta\bar{y}$, $x=\xi(\bar{x})$ $\left(\eta(x)\neq 0, \frac{d\xi}{d\bar{x}}\neq 0\right)$. The resulting equation is of the form

$$\bar{\phi}\bar{y}_{\bar{x}\bar{x}}+\bar{\psi}\bar{y}_{\bar{x}}+\bar{\theta}\bar{y}=0 \quad \text{or} \quad \bar{\phi}\bar{y}_{\bar{x}\bar{x}}+\bar{\psi}\bar{y}_{\bar{x}}+(\lambda\bar{\theta}_1+\bar{\theta}_2)\bar{y}=0,$$

the new boundary conditions for the new interval $\bar{x}_0$, $\bar{x}_1$ being of the form

$$\begin{aligned}
&\bar{\alpha}_1\bar{y}(\bar{x}_0)+\bar{\alpha}_2\bar{y}_{\bar{x}}(\bar{x}_0)+\bar{\alpha}_3\bar{y}(\bar{x}_1)+\bar{\alpha}_4\bar{y}_{\bar{x}}(\bar{x}_1)=0,\\
&\bar{\beta}_1\bar{y}(\bar{x}_0)+\bar{\beta}_2\bar{y}_{\bar{x}}(\bar{x}_0)+\bar{\beta}_3\bar{y}(\bar{x}_1)+\bar{\beta}_4\bar{y}_{\bar{x}}(\bar{x}_1)=0,\\
&e^{\int_{\bar{x}_0}^{\bar{x}_1}\frac{\bar{\psi}}{\bar{\phi}}d\bar{x}}[\bar{\alpha}_1\bar{\beta}_2-\bar{\alpha}_2\bar{\beta}_1]=\bar{\alpha}_3\bar{\beta}_4-\bar{\alpha}_4\bar{\beta}_3,
\end{aligned}$$

as may be shown by computation.* The invariantive property of self-adjointness gives to the results obtained for (1) or (3) a very general character.

§ 1. *Some Properties of Solutions of the Differential Equation.*

In his original memoir Sturm studied the differential equation in the form

$$\frac{d}{dx}\left(K(x,\lambda)\frac{dy}{dx}\right)+\bar{G}(x,\lambda)y=0,$$

where $K>0$ and $\bar{G}$ both depend on a parameter λ. But by a change of variables this may be reduced to the form

$$(py_x)_x+G(x,\lambda)y=0, \tag{4}$$

where $p(x)$ is a positive function independent of λ. We lose nothing in generality by considering this latter equation. Regarded as functions of x the coefficients p, G will be postulated as continuous together with as many derivatives as is desired, while G will be considered as analytic with respect to λ. The usual modifications of the results derived can be written down immediately if less stringent hypotheses are imposed. The trivial solution $y\equiv 0$ will be excluded from the discussion. Some theorems concerning solutions of (4) will now be reviewed.

I. If one zero of a solution of (4) is held fixed, all others are moved nearer to it by a decrease of p or an increase of G. If, for example, $p(x)$ is less than a constant P, and $G(x,\lambda)$ is greater than a constant $\gamma>0$, the zeros of (4) are closer together than those of the equation

$$y_{xx}+gy=0, \quad g=\frac{\gamma}{P},$$

* For the special case of a transformation of the dependent variable only, this computation is given in § 5.

which has a solution $y=\sin\sqrt{g}(x+c)$ with zeros at intervals of $\frac{\pi}{\sqrt{g}}$. By fixing p, and taking G large enough in any interval of x, the zeros of (4) may then be made as close together as desired.

II. The special equation where $G(x,\lambda)$ contains the parameter linearly,

$$(py_x)_x+(q+\lambda k)y=0, \tag{5}$$

has been much discussed. The boundary conditions

$$y(0)=y(1)=0 \tag{6}$$

are of the greatest interest and three cases may be distinguished.

(A) *Orthogonal Case*, when $k(x)\geqq 0$. There is then an infinite number of parameter values $\lambda_m(\lambda_1\leqq\lambda_2\leqq\dots)$ with a limiting point at positive infinity only, for each of which a solution Y_m satisfying (6) exists. The number of zeros of the solution Y_m (including those at $x=0$ and $x=1$) is $m+1$.

(B) *Polar Case*, when $k(x)$ takes on both signs and the integral

$$D(y)=\int_0^1(py_x^2-qy^2)\,dx \tag{7}$$

is positive-definite,* that is, for the given boundary condition (6) $D(y)$ can not take on negative values. There are two sets each of an infinite number of parameter values $0\leqq\lambda_1\leqq\lambda_2\dots$, $0\geqq\lambda_{-1}\geqq\lambda_{-2}\dots$, with limiting points at positive and negative infinity respectively, corresponding to which solutions Y_m, Y_{-m} exist. For both Y_m and Y_{-m} the number of zeros is $m+1$.

(C) *Non-definite Case*, when both of the integrals

$$\int_0^1 ky^2dx,\quad \int_0^1(py_x^2-qy^2)\,dx$$

may take on negative values. This will be discussed in § 4.

III. For the equation (5) and the special boundary conditions (6) certain minimum properties may be stated. In the orthogonal case the minimum of the integral $D(y)$ for those values of y which satisfy (6), and the normalizing and orthogonalizing conditions

$$\int_0^1 ky^2dx=1,\quad \int_0^1 kY_iy\,dx=0,\qquad i=1,2,\dots,m-1, \tag{8}$$

is λ_m, and is furnished by the normalized solution Y_m of (5). In the polar case the minimum of $D(y)$ for those values which satisfy conditions (6) and (8) is λ_m and is furnished by Y_m; the minimum subject to the conditions (6) and

$$\int_0^1 ky^2dx=-1,\quad \int_0^1 kY_{-i}y\,dx=0,\qquad i=1,2,\dots,m-1,$$

* Bôcher has pointed out (*Proceedings International Congress of Mathematics, loc. cit.*, p. 173) that the special case of the polar problem where $q\leqq 0$ can be treated by the method of Sturm. This remark, however, does not apply to the polar case in its most general form.

is $-\lambda_{-m}$ and is furnished by Y_{-m}. In the non-definite case, $D(y)$ can be negative and the minimum (even in the simplest problem ($m=1$)) may not exist. However, in view of the developments of § 4, it would seem probable that by a modification of the discussion, the solution Y_m, for m large enough, may be regarded as furnishing a minimum of a calculus of variations problem.

IV. While the notion of regarding a differential equation directly as the limit of a set of difference equations has been used heuristically since the time of Euler, it was, so far as the author is aware, first made definite and rigorous in a recent paper.* In that paper the problem actually discussed is that of existence theorems for partial differential equations with given boundary conditions. But the same method applies, for example, to equation (1), and the discussion is essentially simpler for the equation in one dimension than for that in two or more. For the sake of simplicity let us confine ourselves to the case (5) (6) and consider the unit interval to be divided into m equal parts, the values of y, p, q, k at the point $\frac{i}{m}(i=0, 1, \ldots, m)$ to be denoted by y_i, p_i, q_i, k_i, and difference equations

$$m^2[p_{i+1}(y_{i+1}-y_i)-p_i(y_i-y_{i-1})]+q_iy_i+\lambda k_iy_i=0 \quad (i=1, 2, \ldots, m-1), \quad (9)$$

to be set up. In order that there be solutions of these equations λ must be one of the $m-1$ roots of the determinant formed from the coefficients. With increase of m the number of points at which y is defined increases, but we can pick out corresponding parameter values and solutions of the various sets of difference equations, and if proper continuity conditions are imposed on the coefficients of the differential equations, it may be shown that the corresponding sets of parameter values approach as a limit a parameter value of (5), and corresponding solutions approach a solution of (5). In this way the infinite set of solutions of the differential equation is obtained. If k has both signs and q is positive and sufficiently large, at least in some part of the interval, some of the λ's and the corresponding y's will be complex.† But in all those cases of equation (5) heretofore treated (the orthogonal and polar), the method of passing to the limit in (9) suffices and gives a simple proof of the fundamental facts. This method can be extended to a treatment of existence theorems for solutions of the equation (4) with more general boundary conditions.

---

* *Transactions of the American Mathematical Society*, Vol. XVIII (1917), p. 489.

† One method of proving this would be by noting that the parameter values and solutions are approximations to those of the differential equation which are shown in § 4 to be complex.

§ 2. *Sufficient Conditions for the Existence of Solutions with an Unlimited Number of Zeros.*

In the consideration of the equation

$$(py_x)_x + G(x, \lambda) y = 0 \tag{10}$$

let us impose the restriction that for finite values of λ the function $G(x, \lambda)$ is limited, and in particular that $G(x, 0)$ is limited. The interesting cases will be covered by one of two hypotheses, which will be justified by the later developments of this section.

HYPOTHESIS A. *For at least one point of the interval, the upper limit of $G(x, \lambda)$ becomes infinite with λ $(\overline{\lim}_{\lambda=+\infty} G(x, \lambda) = +\infty)$, and in such a manner that the number of zeros of the solutions increases without limit with λ.*

The problem treated by Birkhoff* where $\dfrac{\partial G}{\partial \lambda} > 0$ and $G(x, +\infty) = +\infty$ is a special case, and the orthogonal problem (§1, II) is still more special.

HYPOTHESIS B. *For at least one point of the interval $\overline{\lim}_{\lambda=+\infty} G(x, \lambda) = +\infty$; for at least one other, $\overline{\lim}_{\lambda=-\infty} G(x, \lambda) = +\infty$; and in both cases $G(x, \lambda)$ increases in such a manner that the number of zeros of the solutions increases without limit with λ.*

The polar case (§1, II) is included in this hypothesis.

THEOREM I. *In order that there be a set of parameter values λ such that the number of oscillations of the corresponding solutions of (10) be unlimited, it is necessary that in the neighborhood of at least one point $\overline{\lim}_{\lambda=+\infty} G(x, \lambda) = +\infty$ or $\overline{\lim}_{\lambda=-\infty} G(x, \lambda) = +\infty$.*

For, as the number of zeros increases, the length of the smallest interval decreases without limit. The zeros of the equation

$$Py_{xx} + G(x, \lambda) y = 0, \quad P = \text{maximum } p(x) \tag{11}$$

are farther apart than those of (10) (§1, I). To establish the theorem for (10) it is then only necessary to prove it for (11). Let us denote by α, β that pair of consecutive zeros of (11) whose distance is a minimum. In such an interval α, β $[y(\alpha) = y(\beta) = 0]$, y may be taken positive, and since the equation is homogeneous, its solution for all values of λ may be multiplied by a constant so that the maximum is 1. The maximum of y_x must be at least as great as $\dfrac{1}{\beta - \alpha}$ which is the slope of the line joining $(\alpha, 0)$, $(\beta, 1)$. To investi-

---

* *Loc. cit.*

gate the maximum of $-\frac{y_{xx}}{y}$ we note that y_x is zero at some point of the interval and at least as great as $\frac{1}{\beta-\alpha}$ at another. Hence $-y_{xx}$ must be at least as great as $\frac{1}{(\beta-\alpha)^2}$ at some point, and further $\max\left(-\frac{y_{xx}}{y}\right) \geqq \frac{1}{(\beta-\alpha)^2}$. Since for at least one sub-interval the length approaches zero, and since

$$G(x, \lambda) = -\frac{P y_{xx}}{y},$$

it follows that for the interval 0, 1 $\underline{\lim} \max G(x, \lambda) = +\infty$. Since by hypothesis G can become infinite only for $\lambda = \pm\infty$ the theorem may be readily deduced from these results by the usual processes of reasoning.

To show that the necessary condition of *Theorem* I is not sufficient, let us consider the following example: In the interval 0, $\frac{1}{2}-\varepsilon$ we set up the function $y = \frac{2}{\pi}\sin\frac{\pi x}{2}$, and in the interval $\frac{1}{2}+\varepsilon, 1$ the function $y = \frac{2}{\pi}\sin\frac{\pi}{2}(1-x)$; both of these arcs satisfy the equation $y_{xx} + \frac{\pi^2}{4} y = 0$. If continued, they would meet at the point $x = \frac{1}{2}$ with an angle arc tan $2\sqrt{2}$. If in the interval $\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon$ an analytic curve is introduced which is tangent to these two arcs $\left(\text{on one of which } y_x > \frac{1}{\sqrt{2}} \text{ and on the other } y_x < -\frac{1}{\sqrt{2}}\right)$, it must have at some point $-y_{xx} > \frac{1}{\sqrt{2}\varepsilon}$, and if ε is taken small enough, $\max\left(-\frac{y_{xx}}{y}\right)$ is great at pleasure. Hence denoting by $y_{xx} + G(x, \lambda) y = 0$ the differential equation which has for solution the function defined for the interval 0, 1 by this method, and setting $\lambda = \frac{1}{\varepsilon}$, we know that $\lim\limits_{\varepsilon=0} \max G(x, \lambda)$ in the neighborhood of $x = \frac{1}{2}$ increases without limit. On the other hand none of the suite of functions y has any zeros in the interval.

Having shown by this example that the necessary condition of *Theorem* I is not sufficient, let us now deduce a sufficient criterion that when $\lim\limits_{\lambda=\infty} G(x, \lambda) = \infty$, the number of zeros of solutions of (10) be unlimited. A similar discussion can be given for the case $\lim\limits_{\lambda=-\infty} G(x, \lambda) = \infty$. Denoting as before by P the maximum of $p(x)$, and by M an arbitrarily large constant, it is possible to find a λ and an x-interval such that $G(x, \lambda) > MP$ in this interval, whose length will be denoted by ε_M. The number of zeros of the solution $\sin\sqrt{M}(x+c)$ of the equation $y_{xx} + My = 0$ in an interval of length ε_M is not less than the integral

part of $\frac{\sqrt{M}\varepsilon_M}{\pi}$. It follows immediately from § 1, I that the number of zeros of solutions of (10) is not less than that of $\sin\sqrt{M}(x+c)$. Hence the

THEOREM II. *In order that there be N zeros in a solution of* (10) *it is sufficient that one can find an M such that $\frac{\sqrt{M}\varepsilon_M}{\pi}>N$, where ε_M is the length of a sub-interval where $G(x,\lambda)>MP$.*

COROLLARY I. *In order that the number of zeros be unlimited it is sufficient that $\lim\limits_{\lambda=\infty}\varepsilon_M\sqrt{M}=\infty$.*

COROLLARY II. *If throughout any sub-interval of fixed length the value of $G(x,\lambda)$ increases without limit, the number of zeros increases indefinitely.*

However, the sum of the sub-intervals in which $G>M$ may remain above a positive constant without compelling the number of zeros to increase with M. For example, one can set up a function which has no zeros except at $x=0$ and $x=1$ and which, except in the neighborhood of these points, oscillates between $y=\frac{1}{2}$ and $y=\frac{3}{4}$, the number of oscillations increasing indefinitely with λ. Moreover, by taking the oscillations frequent enough one can have $G(x,\lambda)>MP$ in portions which total at least one-quarter (or any other proper fraction) of the interval, the great curvature downward in these sub-intervals being counterbalanced by curvature upward in those where $G(x,\lambda)$ has the opposite sign.

On the other hand, we have seen that the number of zeros of the solution $\sin\sqrt{M}(x+c)$ of the equation $y_{xx}+My=0$ is not less than the integral part of $\frac{\sqrt{M}\varepsilon_M}{\pi}$ where ε_M is the length of the interval. If this interval is divided into two parts ε_1, ε_2 for which there are solutions $\sin\sqrt{M}(x+c_1)$, $\sin\sqrt{M}(x+c_2)$ respectively, the number of zeros can not be reduced by more than one. For, the sum of the integral parts of $\frac{\sqrt{M}\varepsilon_1}{\pi}$ and $\frac{\sqrt{M}\varepsilon_2}{\pi}$ cannot differ by more than one from the integral part of $\frac{\sqrt{M}(\varepsilon_1+\varepsilon_2)}{\pi}$. It follows in the same way that if the interval is divided into n_1 parts the number of zeros of the solutions can not be decreased by more than n_1-1. But in the intervals where $G(x,\lambda)>MP$ the zeros of the solutions of (10) must be at least as many as the minimum number for $y_{xx}+My=0$.

THEOREM III. *In order that the solution of* (10) *have N zeros it is sufficient that one can find an M such that $\frac{\sqrt{M}\eta_M}{\pi}-n_1+1>N$, where η_M denotes the sum of the lengths of the n_1 intervals in which $\frac{G}{\max p(x)}>M$.*

§ 3. *Behavior of the Zeros with Monotone Change of λ.*

Let us fix the zero at the left-hand end of the interval 0, 1, denote such a solution by Y $[Y(0)=0]$, and investigate what happens to the other zeros as λ increases. It is convenient to think of $G(x, \lambda)$ as being defined for values $x>1$. Since the coefficients of (10) are continuous, the zeros move continuously, and since we can not have both $y(x)=0$ and $y'(x)=0$ without having $y\equiv0$, the zeros can not coalesce and then disappear. By differentiating (10) with regard to λ we have

$$\frac{\partial (py_x)_x}{\partial\lambda}+G\frac{\partial y}{\partial\lambda}+\frac{\partial G}{\partial\lambda}y=0, \tag{12}$$

and on multiplication of this equation by $-y$ and of (10) by $\frac{\partial y}{\partial\lambda}$, addition and integration from x_1 to x_2 we get the fundamental formula *

$$\left[py_x\frac{\partial y}{\partial\lambda}\right]_{x_1}^{x_2}-\left[py\frac{\partial y_x}{\partial\lambda}\right]_{x_1}^{x_2}=\int_{x_1}^{x_2}\frac{\partial G}{\partial\lambda}y^2dx. \tag{13}$$

For the particular solution $Y(x)$ we have $Y(0)=0, \frac{\partial Y(0)}{\partial\lambda}$, and if α is another zero of Y the formula becomes

$$p(\alpha)Y_x(\alpha)\frac{\partial Y}{\partial\lambda}(\alpha)=\int_0^\alpha\frac{\partial G}{\partial\lambda}Y^2dx. \tag{13'}$$

If the integral on the right is positive, the signs of $Y_x(\alpha)$ and $\frac{\partial Y(\alpha)}{\partial\lambda}$ are the same. Hence when $Y_x(\alpha)$ is negative, $Y(\alpha)$ is decreasing with increase of λ, and this zero of Y is moving to the left; when $Y_x(\alpha)$ is positive, $Y(\alpha)$ is increasing and the zero is again moving to the left. In the same way it may it may be shown that all zeros for which the integral on the right of (13′) is negative, move to the right. When the integral is zero, further investigation is needed. We have now proved

---

*If p is also a function of λ this formula becomes

$$\left[py_x\frac{\partial y}{\partial\lambda}\right]_{x_1}^{x_2}-\left[py\frac{\partial y_x}{\partial\lambda}\right]_{x_1}^{x_2}-\left[\frac{\partial p}{\partial\lambda}y_xy\right]_{x_1}^{x_2}=\int_{x_1}^{x_2}\frac{\partial G}{\partial\lambda}y^2dx-\int_{x_1}^{x_2}\frac{\partial p}{\partial\lambda}y_x^2dx.$$

In terms of this hypothesis results analogous to all the succeeding theorems of this paper may be written down. Since, however, by a transformation of variables p can be made independent of λ we shall, for the sake of simplicity, confine ourselves to the problem proposed.

If at $x=0$ the condition $y_x(0)+\sigma y(0)=0$ is imposed where σ is a constant, then $\frac{\partial y_x(0)}{\partial\lambda}+\sigma\frac{\partial y(0)}{\partial\lambda}=0$, and this formula (13) can be written

$$-y^2p\frac{d}{dx}\left(\frac{\frac{\partial y}{\partial\lambda}}{y}\right)\equiv p\left[y_x\frac{\partial y}{\partial\lambda}-y\frac{\partial y_x}{\partial\lambda}\right]=\int_0^x\frac{\partial G}{\partial\lambda}y^2dx.$$

Hence, in order that the roots of a solution y and the roots of $\frac{\partial y}{\partial\lambda}$ alternate it is sufficient that $\int_0^x\frac{\partial G}{\partial\lambda}y^2dx$ have one sign.

THEOREM IV. *If $Y(0)=0$ and if λ is a parameter value for which there are subsequent zeros of Y at $\alpha_1, \ldots, \alpha_n$ then with increase of λ a zero, α_i, moves to the left or right according as*

$$\int_0^{\alpha_i} \frac{\partial G(x,\lambda)}{\partial\lambda} Y^2(x,\lambda)\,dx$$

is positive or negative.

The argument used in deducing *Theorem* IV is still valid if in place of a zero of y we introduce a zero of y_x.

THEOREM IV A. *If $y(0)=0$ or $y_x(0)=0$ and if λ is a parameter value such that $y(\alpha)=0$ or $y_x(\alpha)=0$, then with increase of λ the zero α moves to the left or right according as*

$$\int_0^{\alpha} \frac{\partial G(x,\lambda)}{\partial\lambda} y^2(x,\lambda)\,dx$$

is positive or negative.

COROLLARY. *In order that λ be a multiple characteristic number and $Y(x,\lambda)$ a multiple solution of the problem* (10) (6) *it is necessary* that*

$$\int_0^1 \frac{\partial G(x,\lambda)}{\partial\lambda} Y^2(x,\lambda)\,dx=0.$$

G can be so chosen as a function of λ that with increasing λ the integral $\int_0^1 \frac{\partial G}{\partial\lambda} Y^2 dx$ is alternately positive and negative, and since the corresponding intervals of λ may be small at pleasure, we see that a zero may pass back and forth through $x=1$ an infinite number of times. There may then be an infinite number of λ's for which exist solutions of (10) (6) with a fixed number of zeros.

As λ runs through its interval there will be at least one value for which the number of zeros of $Y(x,\lambda)$ is a minimum. For both the orthogonal and polar cases this minimum number of oscillations is zero, but in general this will not be the case. If for $\lambda=\lambda'$ there is a minimum number n_1 of oscillations, then from the principle of continuity we can argue that for $n>n_1$ there is under *Hypothesis A* (§ 2) at least one value of λ for which exists a solution of (10) vanishing at $x=0$ and $x=1$, and oscillating n times (having n zeros) in the interval, while under *Hypothesis B* there are at least two.

One fundamental problem for investigation is the determination of sufficient conditions that in a given interval of λ (which may include infinity) there is *only one* solution of (10) (6) oscillating a given number of times; in

---

* This condition may be shown to be sufficient also.

other words, sufficient conditions that the zeros of $Y(x)$ pass through the point $x=1$ in one direction only as λ increases (or decreases). Another problem is that of determining when there exist precisely two solutions oscillating a given number of times.

THEOREM V. *If for values of λ in an interval (which may include infinity) there are solutions of* (10) (6) *for which the minimum and maximum number of oscillations are n_1 and n_2 (n_2 may be infinite), respectively, and if the value of $\frac{\partial G}{\partial \lambda}$ is equal to or greater than $\phi(\lambda)G(x,\lambda)$, where ϕ is a positive function of λ, then there is one and only one solution oscillating n times ($n_1 \leqq n \leqq n_2$) in the interval $0 \leqq x \leqq 1$.*

For, we have on multiplying (10) by Y, and integrating under the boundary conditions (6),

$$\int_0^1 pY_x^2 dx = \int_0^1 GY^2 dx. \tag{14}$$

From the hypothesis we may then write

$$\int_0^1 \frac{\partial G}{\partial \lambda} Y^2 dx \geqq \phi(\lambda) \int_0^1 GY^2 dx = \phi(\lambda) \int_0^1 pY_x^2 dx > 0$$

and since this holds for all λ, *Theorem* IV gives the desired result.

The theorem just proved fits the case where with unlimited increase of λ the maximum of G increases without limit. For the case where λ approaches $-\infty$, we observe from *Theorem* IV that a sufficient condition for a similar theorem is that $\int_0^{\prime} \frac{\partial G}{\partial \lambda} Y^2 dx$ be negative. To insure this, it is sufficient to take $\frac{\partial G}{\partial \lambda} < -\psi G$, where $\psi(\lambda)$ is a positive function. Then

$$\int \frac{\partial G}{\partial \lambda} Y^2 dx < -\psi(\lambda) \int GY^2 dx = -\psi(\lambda) \int pY_x^2 dx < 0.$$

Combining this result with *Theorem* V we may state the following:

THEOREM VI. *If for two values λ', λ'' ($\lambda' \geqq \lambda''$) of λ there are n_1 zeros of the solution Y within the interval* 0, 1, *and if for $\lambda > \lambda'$, $\frac{\partial G}{\partial \lambda} \geqq \phi_1(\lambda) G(x,\lambda)$, and for $\lambda < \lambda''$, $\frac{\partial G}{\partial \lambda} \leqq -\phi_2(\lambda) G(x,\lambda)$ where ϕ_1, ϕ_2 are positive functions defined in the intervals λ', $+\infty$ and $-\infty$, λ'', respectively, then under Hypothesis B* (§ 2) *for $n \geqq n_1$ there are two and only two values of λ for which there is a solution $Y(x,\lambda)$ of* (10) (6) *with n zeros.*

In the polar problem we have the special case of the hypothesis of this theorem where $\lambda'=\lambda''=0$, $n_1=0$, $G=q+\lambda k$, $\frac{\partial G}{\partial \lambda}=k$, $\phi_1=\frac{1}{\lambda}>0$, $-\phi_2=\frac{1}{\lambda}<0$. A more general special case of *Theorem* VI is discussed in § 4.

Since formula (14) is valid also when

$$y(0)=y_x(1)=0, \text{ or } y_x(0)=y(1)=0, \text{ or } y_x(0)=y_x(1)=0,$$

theorems analogous to VI can be written down for any of these boundary conditions.

§ 4. *The Non-Definite Case.*

With regard to applications, the differential equation

$$(py_x)_x+(q+\lambda k)y=0, \tag{15}$$

with boundary conditions

$$y(0)=y(1)=0, \tag{16}$$

is the most important special type of (10). The problem where one of the integrals

$$D(y, 0, 1)=\int_0^1 (py_x^2-qy^2)\,dx, \quad \int_0^1 ky^2 dx$$

is definite, has been studied in great detail. Let us first investigate some properties of the remaining *non-definite case* where both these integrals may take on negative values and later deduce some analogous results for the more general equation (10). In §§ 6–8 these results will be extended to cover the more general boundary condition (2).

For $q \leqq 0$ the integral $D(y, 0, 1)$ can not be negative. If q is positive in at least a part of the interval 0, 1, the minimum of the integral $\int_0^1 py_x^2 dx$ under the conditions (16) and $\int_0^1 qy^2 dx=1$ is (§ 1, III) the smallest parameter value λ of the equation $(py_x)_x+\lambda qy=0$, and when $\frac{q}{\min p} \leqq \pi^2$, this in turn is (§ 1, I) at least as great as the smallest of the equation $y_{xx}+\lambda\pi^2 y=0$ under the same boundary conditions (16). Since for the latter equation $\lambda_1=1$ and $Y_1=c \sin \pi x$, it follows that λ is greater than unity. Hence $\int_0^1 py_x^2 dx=\lambda$ is greater than $\int_0^1 qy^2 dx=1$ and $D(y, 0, 1)$ is positive.

In the non-definite case one must therefore have $\frac{q}{\min p}>\pi^2$ *in at least a part of the interval,* and this assumption we shall now make. But we note here that in a sub-interval α_1, α_2 such that $y(\alpha_1)=y(\alpha_2)=0$, any value of q for which $\frac{q}{\min p}<\frac{\pi^2}{(\alpha_2-\alpha_1)^2}$, makes the integral $\int_{\alpha_1}^{\alpha_2}(py_x^2-qy^2)\,dx$ definite, as is

readily shown by a comparison argument like that above. Hence, for given p and q, if throughout the interval 0, 1 the zeros can be taken close enough together (as *e.g.* may certainly be done where λ has the same sign as $k(x)$ and is taken large enough in absolute value, cf. § 1, I), the integral for each sub-interval is definite, and hence $D(y, 0, 1)$ is positive. *This is also true under the more general boundary conditions* (2).

On the other hand from the formula

$$D(y, \alpha_1, \alpha_2) \equiv \int_{\alpha_1}^{\alpha_2} (py_x^2 - qy^2)\,dx = \lambda \int_{\alpha_1}^{\alpha_2} ky^2 dx, \tag{17}$$

which is obtained by multiplying (15) by y and integrating under the conditions $y(\alpha_1) = y(\alpha_2) = 0$, we see that if λ is positive and k negative in the interval, then $D(y, \alpha_1, \alpha_2)$ is negative; it is, however, still possible that for the larger interval 0, 1 the integral $D(y, 0, 1)$ be positive. Were this the case (as will later be established provided λ is large enough) we could deduce from *Theorem IV* and the formula

$$D(y, 0, 1) \equiv \int_0^1 (py_x^2 - qy^2)\,dx = \lambda \int_0^1 ky^2 dx, \quad y(0) = y(1) = 0, \tag{18}$$

certain facts in regard to the movement of the subsequent zeros of particular solutions Y of (15) which vanish at $x=0$. Before proceeding to a detailed discussion of this matter let us compare some possibilities for the orthogonal, polar and non-definite cases.

In the orthogonal case let us consider all λ greater than some finite number chosen less than the smallest characteristic (which may be positive or negative). As λ increases, the subsequent zeros of any solution Y of (15) $[Y(0) = 0]$ move to the left, new ones being added to the interval 0, 1. For the smallest value of λ there are no zeros within the interval and when λ passes through a parameter value λ_n the n-th zero enters. In the polar case there are two ranges of values of λ; for the range which extends from zero to $+\infty$, a result precisely like that of the orthogonal case may be stated; for the range which extends from $-\infty$ to zero, a decrease of λ causes subsequent zeros to move to the left, there being no zero of $Y(x, 0)$ present in the interval while the n-th enters for $\lambda = \lambda_{-n}$. In the non-definite case we know that the results must be quite different and we may expect that the march of the zeros will not be monotone with λ. In fact, *there may be a range of values of* λ $(L_1 \leqq \lambda \leqq L_2)$ *such that as* λ *increases the number of zeros first decreases, then increases, then decreases and finally increases, the minimum number being a positive integer.**

* A reference to the proof of this last fact is given in the Introduction.

As an example let us consider the equation

$$y_{xx}+[(100\pi)^2+\lambda\varkappa)y=0, \quad y(0)=y(1)=0,$$

where $\varkappa$ is equal -1 in the sub-interval $0, \frac{1}{2}$ and equal 1 in the remainder of the interval. Such a function $\varkappa(x)$ may be approximated by an analytic function $k(x)$ for which the corresponding equation has solutions with similar properties. When $\lambda=0$ the solution $y=\sin 100\pi x$ has 100 zeros in the interval. When $\lambda=(100\pi)^2$, we have the equation $y_{xx}=0$ in $0, \frac{1}{2}$ and $y_{xx}+2(100\pi)^2y=0$ elsewhere, the solutions being respectively $y=cx$ ($c=$const.), $y=\sin 100\sqrt{2}\pi x$. There are are then approximately $50\sqrt{2}$ zeros. For $\lambda=-(100\pi)^2$ there are evidently the same number of zeros. More generally, when λ is in the interval $-(100\pi)^2$, $(100\pi)^2$ the number of zeros is approximately

$$\frac{1}{2\pi}[\sqrt{(100\pi)^2+\lambda}+\sqrt{(100\pi)^2-\lambda}],$$

a function which has its maximum for $\lambda=0$ and *decreases* with increase of $|\lambda|$. On the other hand when $|\lambda|$ is greater than $(100\pi)^2$ the number of zeros is approximately $\frac{1}{2\pi}\sqrt{(100\pi)^2+|\lambda|}$, which *increases* with $|\lambda|$. The minimum number of zeros is then approximately $50\sqrt{2}$ and occurs for $\lambda=(100\pi)^2$ and $-(100\pi)^2$.* *For values of n between $50\sqrt{2}$ and 100 there are four parameters corresponding, to which exist solutions oscillating n times.*

Returning now to the discussion of the sign of $D(y, 0, 1)$ we can prove

LEMMA I. *If in an interval a, b, the function k is positive, except perhaps at the end points, then λ can be taken so large that for solutions of* (15),

$$D(y, a, b)\equiv\int_a^b (py_x^2-qy^2)\,dx>0.$$

For, by taking λ large enough, $q+\lambda k$ can be made as large as desired, except perhaps at the end points. Hence the zeros are as thickly strewn throughout the interval a, b as desired (§ 1, I). We note that in any sub-interval β_1, β_2 of a, b, in which there is a zero of y and for which the maximum of $|y|$ is M, the minimum value of $\int_{\beta_1}^{\beta_2} y_x^2 dx$ is taken on when y is a straight line joining $(\beta_1, 0)$ and $(\beta_2, \pm M)$. This minimum value is $\frac{M^2}{\beta_2-\beta_1}$. Since on the other hand $\int_{\beta_1}^{\beta_2} qy^2dx<(\beta_2-\beta_1)M^2 \max q$, the value of $D(y, \beta_1, \beta_2)$ is greater than $M^2\left[\frac{\min p}{\beta_2-\beta_1}-(\beta_2-\beta_1)\max q\right]$: hence when $(\beta_2-\beta_1)^2$ is made less than

---

* By reference to *Theorem* IV it follows from this discussion that in the two intervals which are approximately $-(100\pi)^2, 0$; $(100\pi)^2, +\infty$ the value of $\int_0^1 \varkappa y^2 dx$ must be positive or $\int_0^{\frac{1}{2}} y^2dx<\int_{\frac{1}{2}}^1 y^2dx$, while in the intervals which are approximately $-\infty, -(100\pi)^2$; $0, (100\pi)^2$, $\int_0^{\frac{1}{2}} y^2dx>\int^1 y^2dx$.

$\frac{\min p}{\max q}$ this integral is definite. It follows that since the first and last zeros of $y(x)$ are as close to the ends of the interval a, b as is desired, those portions of the integral arising from these two end sub-intervals can be made positive. For any interval α_1, α_2 of oscillation $[y(\alpha_1)=y(\alpha_2)=0]$ it follows at once from (17) that when k and λ are positive $D(y, \alpha_1, \alpha_2)>0$. Combining these results we have the lemma.

It follows in the same way that if $k \leq 0$ and $k \not\equiv 0$ in an interval a, b, then λ can be taken so large negatively that for solutions of (15) the integral $D(y, a, b)$ is positive.

Coming now to an interval c, d in which k is negative, except perhaps at c and d, it is possible in an interval $c+\eta, d-\eta$ ($\eta>0$ arbitrarily small) to take M' arbitrarily large and then choose λ so large that $-(q+\lambda k)>M'$ or

$$|(py_x)_x|=|-(q+\lambda k)y|>M'|y|, \quad [(py_x)_x>M'y \text{ if } y>0]. \tag{19}$$

The arc $y(x)$ may be shown to be sharply concave away from the x-axis.* It is possible to prove the following:

Lemma II. *If in an interval c, d the function k is negative, except perhaps at the end points, then λ can be taken so large that for a solution of* (15)

$$D(y, c, d)\equiv\int_c^d (py_x^2-qy^2)\,dx>0.$$

This will be proved by showing that on taking λ great enough it is possible to insure that $\frac{y_x^2}{y^2}$ is great at pleasure except perhaps in sub-intervals whose length decreases indefinitely with $\frac{1}{\lambda}$. Since in the interval $c+\eta, d-\eta$ the curve is concave upward for positive y and concave downward for negative y, there can not be more than one zero. The discussion may be separated into two parts according as $y(x)$ has a zero or not.

If $y(\gamma)=0$, then without loss of generality it may be assumed that within the interval $\gamma, d-\eta$, y is positive and hence, since $y_{xx}>0$, that y_x is positive. Multiplying the inequality (19) by $2py_x$ and integrating we get

$$\int_\gamma^x 2py_x(py_x)_x\,dx>M'\int_\gamma^x 2ypy_x\,dx,$$

and hence

$$p^2(x)y_x^2(x)\geq p^2(x)y_x^2(x)-p^2(\gamma)y_x^2(\gamma)\geq M'[\min p]\int_\gamma^x 2yy_x\,dx$$
$$=M'y^2(x)\min p, \tag{20}$$

* This is at once evident if the equation is taken in the form $\bar{y}_{\bar{x}\bar{x}}+(\bar{q}+\lambda\bar{k})\bar{y}=0$ to which it may be reduced by a transformation.

for all x in the interval γ, $d-\eta$. Similar reasoning establishes the same result in the interval $c+\eta$, γ. Hence, for the case that y vanishes, we have throughout the interval $c+\eta$, $d-\eta$ the inequality $\frac{y_x^2}{y^2} > \frac{M' \min p}{p^2}$, which as noted above, is sufficient to establish the theorem.

If on the other hand y has no zero in the interval, let us take the function positive and denote by γ the point at which it takes on its minimum. It may be assumed that γ is not at the right-hand end of the interval; that special case could be treated in an analogous fashion. By multiplying y by a constant, $y(\gamma)$ can be made equal 1, while the sign of the integral $D(y, c, d)$ is not altered. Since $y_x(\gamma) \geqq 0$, we have $y>1$ in the neighborhood of γ; in fact, for any arbitrarily small but fixed η we can, by taking M' large enough, have $y(\gamma+\eta)>2$. For, since $y \geqq 1$ it follows from the inequality (19) that

$$py_x \geqq py_x - py_x(\gamma) = \int_\gamma^x (py_x)_x dx \geqq M' \int_\gamma^x y dx = M'(x-\gamma).$$

Integrating again,

$$(y-1)\max p = \max p \int_\gamma^x y_x dx \geqq \int_\gamma^x py_x dx \geqq M' \int_\gamma^x (x-\gamma)\, dx = \frac{M'(x-\gamma)^2}{2}.$$

By taking $x-\gamma \geqq \eta$ and choosing $\frac{M'\eta^2}{2\max p} > 1$ we have $y-1>1$ or $y>2$. From this it follows that on multiplying the inequality (19) by $2py_x$ and integrating we get by a process similar to that used in (20) the formula

$$p^2(x) y_x^2(x) > M' \int_\gamma^x 2ypy_x dx \geqq M'[y^2(x)-1] \min p > \frac{3M'}{4} y^2(x) \min p.$$

Hence in the interval $\gamma+\eta$, $d-\eta$ we have $\frac{y_x^2}{y^2} > \frac{3M'}{4}\frac{\min p}{p^2}$. And since in the interval $c+\eta$, $\gamma-\eta$ a similar inequality may be obtained, this completes the proof of the lemma.

It follows in the same way that if $k>0$, except perhaps at the end points of the interval c, d, then λ can be taken so large negatively that for solutions of (15) the integral $D(y, c, d)$ is positive.

We have then proved by these lemmas that in any sub-interval in which k has one sign, $D(y)$ can be made positive by taking λ large enough positively and also by taking λ large enough negatively. It follows that $D(y, 0, 1)$ would be positive, and from the formula

$$\lambda \int_0^1 ky^2 dx = \int_0^1 (py_x^2 - qy^2)\, dx \equiv D(y, 0, 1) > 0$$

and *Theorem* IV, that for $|\lambda|$ large enough the zeros of Y move to the left with increase of $|\lambda|$. We can then enunciate

THEOREM VII. *There exists an integer n_2 such that for $n \geqq n_2$ there are precisely two solutions of* (15) (16) *oscillating n times.*

If proper restrictions are imposed on $G(x, \lambda)$, a theorem similar to the preceding may be proved for the general equation

$$(py_x)_x + G(x, \lambda) y = 0. \tag{10}$$

The proof used in *Theorems* VI and VII, and *Lemmas* I and II is valid also under the following hypotheses on a function $G'(x, \lambda)$ obtained by subtracting from $G(x, \lambda)$ a function $q(x) \cdot [G'(x, \lambda) \equiv G(x, \lambda) - q(x)]$.

(α) In one or more sub-intervals of $0, 1$, $G'(x, \lambda)$ is a monotone increasing function of λ such that $\lim\limits_{\lambda=\infty} G'(x, \lambda) \geqq 0$ and for at least a part of each sub-interval $\lim\limits_{\lambda=\infty} G(x, \lambda) = +\infty$.

(β) In the remaining sub-interval or sub-intervals of $0, 1$, $G'(x, \lambda)$ is a decreasing function of λ such that $\lim\limits_{\lambda=-\infty} G'(x, \lambda) \leqq 0$, and for at least a part of each sub-interval $\lim\limits_{\lambda=-\infty} G'(x, \lambda) = -\infty$.

(γ) For values λ', λ'' of λ there are n_1 zeros of the solution Y within the interval $0, 1$ and for $\lambda > \lambda'$,

$$\frac{\partial G'}{\partial \lambda} \geqq \phi_1(\lambda) G'(x, \lambda), \text{ and for } \lambda < \lambda'', \frac{\partial G'}{\partial \lambda} \leqq -\phi_2(\lambda) G'(x, \lambda),$$

where ϕ_1, ϕ_2 are positive functions defined in the intervals λ', $+\infty$ and $-\infty$, λ'' respectively.

THEOREM VIII. *Under the conditions* (α) (β) (γ) *there exists an integer n_2 such that for $n > n_2$ there are precisely two solutions of* (10) (16) *oscillating n times.*

When the parameter value and corresponding solution of (15) (16) are complex, let us set $\lambda \equiv \sigma + i\tau$, $y \equiv u + iv$. The differential equation resolves itself into the two following:

$$(pu_x)_x + qu + \sigma ku - \tau kv = 0, \quad u(0) = u(1) = 0, \tag{21}$$

$$(pv_x)_x + qv + \sigma kv + \tau ku = 0, \quad v(0) = v(1) = 0. \tag{22}$$

Corresponding to $\lambda \equiv \sigma - i\tau$ there is then a solution $y \equiv u - iv$. On multiplying (21) and (22) in the first place by u and v, and in the second by v and $-u$ respectively, adding and integrating, one obtains the two formulae

$$\int_0^1 [p(u_x^2 + v_x^2) - q(u^2 + v^2)]\,dx = 0, \quad \int_0^1 k(u^2 + v^2)\,dx = 0. \tag{23}$$

More generally one obtains by this process the formula

$$[p(u_x v - v_x u)]_{x=\alpha}^{x=\beta} = \tau \int_\alpha^\beta k(u^2+v^2)\,dx. \tag{24}$$

The orthogonality of two solutions ($\int_0^1 kY_n Y_m dx = 0$, $m \neq n$) holds as well for complex as for real characteristic numbers λ_n, λ_m. Hence by separation into real and imaginary parts we get the

THEOREM IX. *If* $Y_m(x, \lambda_m) \equiv u_m + iv_m$, $Y_n \equiv u_n + iv_n$ *are two solutions of* (15) (16), *then*

$$\int_0^1 k(u_m u_n - v_m v_n)\,dx = 0, \quad \int_0^1 k(u_m v_n + u_n v_m)\,dx = 0.$$

Theorems of other types may be derived of which the following is an example:

THEOREM X. *If in the interval* 0, 1, *the function* $k(x)$ *changes sign once only, then the roots of the real and imaginary parts* u_n, v_n *of the solution* $Y_n \equiv u_n + iv_n$ *of* (15) (16) *separate one another.*

For, formula (24) may be written

$$p(u_x v - v_x u) = \int_0^x k(u^2+v^2)\,dx, \text{ or } p\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{1}{v^2}\int_0^x k(u^2+v^2)\,dx.$$

From the hypothesis that k changes sign but once it follows that the integral can not vanish within the interval, and since $\frac{d\left(\frac{u}{v}\right)}{dx} > 0$ except at the end points, $\frac{u}{v}$ is a monotone function and the theorem follows at once. We may note further that under the hypotheses of the theorem neither u nor v can gain or lose a zero at a point within the interval. For, were there a zero lost or gained at $x = a$, both $y(a)$ and $y_x(a)$ would vanish, and formula (24) could be written $\int_0^a k(u^2+v^2)\,dx = 0$, which would give a contradiction.

§5. *Reduction of the General Boundary Conditions to Normal Forms.*

In the preceding sections the discussion has dealt mainly with the simple boundary conditions $y(0)y_x(0) = y(1)y_x(1) = 0$. To facilitate the discussion of the most general boundary conditions it is desirable to obtain normal forms.* The linear self-adjoint equation of the second order will be taken in the form

$$(\pi(x)u_x(x))_x + \Gamma(x, \lambda)u(x) = 0, \quad \pi(x) > 0, \tag{25}$$

* The classification used in this section follows that of Hilbert and Haupt, *loc. cit.* The geometrical form into which the transformation is thrown was suggested by my colleague, Prof. H. P. Manning.

and if the linearly independent boundary conditions

$$\left.\begin{aligned}\alpha_1 u(0)+\alpha_2 u_x(0)+\alpha_3 u(1)+\alpha_4 u_x(1)=0,\\ \beta_1 u(0)+\beta_2 u_x(0)+\beta_3 u(1)+\beta_4 u_x(1)=0,\end{aligned}\right\} \tag{26}$$

are to be self-adjoint, the condition

$$\pi(0)\begin{vmatrix}\bar{u}(0) & \bar{u}_x(0)\\ \bar{\bar{u}}(0) & \bar{\bar{u}}_x(0)\end{vmatrix}=\pi(1)\begin{vmatrix}\bar{u}(1) & \bar{u}_x(1)\\ \bar{\bar{u}}(1) & \bar{\bar{u}}_x(1)\end{vmatrix} \tag{27}$$

must be imposed, where $\bar{u}$, $\bar{\bar{u}}$ are any functions with continuous derivatives which satisfy (26). We have the relation

$$\begin{vmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{vmatrix}\begin{vmatrix}\bar{u}(0) & \bar{u}_x(0)\\ \bar{\bar{u}}(0) & \bar{\bar{u}}_x(0)\end{vmatrix}=\begin{vmatrix}\alpha_3 & \alpha_4\\ \beta_3 & \beta_4\end{vmatrix}\begin{vmatrix}\bar{u}(1) & \bar{u}_x(1)\\ \bar{\bar{u}}(1) & \bar{\bar{u}}_x(1)\end{vmatrix}, \tag{28}$$

as may be seen by applying the usual rule for multiplying determinants and noting from (26) that each of the four elements of one product determinant is the negative of the corresponding element of the other. Let us denote by B_{ij} the determinant formed by taking the i-th and j-th columns of the matrix

$$\begin{Vmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\end{Vmatrix}.$$

Considering the two determinants of the u's as the variables, we have from the theory of linear equations that a necessary and sufficient condition for the solution of (27) and (28) is that

$$\pi(1)B_{12}-\pi(0)B_{34}=0. \tag{29}$$

Hence B_{12} and B_{34} are simultaneously zero or different from zero.

Let us subject the dependent variable to the transformation

$$u(x)=\eta(x)y(x), \quad \eta(x)\neq 0, \tag{30}$$

by which the number of zeros of the solution remains unaltered. The equation (25) takes on the self-adjoint form

$$(py_x)_x+G(x,\lambda)y=0, \quad p=\pi\eta^2>0, \quad G(x,\lambda)=\pi\eta\eta_{xx}+\pi_x\eta\eta_x+\Gamma\eta^2, \tag{31}$$

and the conditions (26) are replaced by a similar set

$$\left.\begin{aligned}\gamma_1 y(0)+\gamma_2 y_x(0)+\gamma_3 y(1)+\gamma_4 y_x(1)=0,\\ \delta_1 y(0)+\delta_2 y_x(0)+\delta_3 y(1)+\delta_4 y_x(1)=0.\end{aligned}\right\} \tag{32}$$

We have at once from (31) the important formula

$$\int_0^1\frac{\partial\Gamma}{\partial\lambda}u^2dx=\int_0^1\frac{\partial G}{\partial\lambda}y^2dx. \tag{33}$$

Writing u_1, u_2, u_3, u_4, for $u(0)$, $u_x(0)$, $u(1)$, $u_x(1)$, and using a similar notation for the η's and y's we can interpret our problem in relation to the tetraedron of reference as the investigation of the line (26) subject to the condition (27) or (29). If $B_{34}=0$, the u_3 and u_4 can be eliminated from (26) at the same time, giving

$$B_{13}u_1+B_{23}u_2=0 \text{ or } B_{14}u_1-B_{42}u_2=0, \tag{34}$$

either of these equations representing the plane determined by the given line and the edge $12(u_1=0,\ u_2=0)$. Hence the line (26) intersects this edge of the tetraedron. But if $B_{34}=0$ we have from (29) $B_{12}=0$, and the straight line intersects also the edge 34. The equation of the plane determined by the given line and the edge 34 can be written

$$B_{13}u_3+B_{14}u_2=0 \text{ or } B_{23}u_3-B_{42}u_2=0. \tag{35}$$

From (34) and (35) we see that under the hypothesis $B_{12}=B_{34}=0$ the boundary conditions may be written

Case I. $$\sigma u_1+u_2=0, \quad \tau u_3+u_4=0, \tag{36}$$

where

$$\sigma=\frac{B_{13}}{B_{23}}=-\frac{B_{14}}{B_{42}}, \quad \tau=\frac{B_{13}}{B_{14}}=-\frac{B_{23}}{B_{42}}.$$

The parameters σ and τ may have zero or infinite values when the line (26) lies in a face of the tetraedron.

In general, when any two B's with complementary indices are zero, the line intersects two opposite edges of the tetraedron and the conditions (26) may be reduced to a normal form similar to (36).

The transformation (30) takes the form

$$u_1=\eta_1y_1, \quad u_2=\eta_2y_1+\eta_1y_2, \quad u_3=\eta_3y_3, \quad u_4=\eta_4y_3+\eta_3y_4, \quad \eta_1\neq 0,\ \eta_2\neq 0, \tag{37}$$

and corresponds to a rotation of the face 2 of the tetraedron about the edge 12, and of the face 4 about the edge 34, leaving the faces 1 and 2 and the edges 12, 13 and 34 unchanged. If we denote by A_{ij} the determinants of the matrix of the coefficients γ_i, δ_i of (32) we can read their values from the identity

$$\left|\begin{matrix}\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4\\ \delta_1 & \delta_2 & \delta_3 & \delta_4\end{matrix}\right| \equiv \left|\begin{matrix}\eta_1\alpha_1+\eta_2\alpha_2 & \eta_1\alpha_2 & \eta_3\alpha_3+\eta_4\alpha_4 & \eta_3\alpha_4\\ \eta_1\beta_1+\eta_2\beta_2 & \eta_1\beta_2 & \eta_3\beta_3+\eta_4\beta_4 & \eta_3\beta_4\end{matrix}\right|;$$

thus $A_{12}=\eta_1^2B_{12}$, $A_{34}=\eta_3^2B_{34}$, etc. As may be seen from these formulae and (31), we have corresponding to (27) or (29), the relation

$$p(1)A_{12}-p(0)A_{34}=0 \tag{38}$$

which, when added to (32), makes these boundary conditions self-adjoint.

The discussion of the boundary conditions under the transformation may be sub-divided as follows:

I. If the given line intersects the edges 12 and 34 ($B_{34}=B_{12}=0$), the same will be true after the transformation since $A_{34}=A_{12}=0$. To reduce the condition to the normal form (36) (which we shall in general use) the transformation is superfluous. This case is called the sturmian.

From the formulae (37) it follows at once that the conditions (36) can by proper choice of $\eta_1, \eta_2, \eta_3, \eta_4$ be reduced to one of the simpler forms

$$\text{(a) } y_1=y_3=0; \quad \text{(b) } y_1=y_4=0; \quad \text{(c) } y_2=y_3=0; \quad \text{(d) } y_2=y_4=0, \tag{39}$$

these corresponding geometrically to the cases where the given line coincides respectively with the edge 13, the new edge 14, the new edge 23 or the new edge 24.

II. If the line does not intersect the edges 12, 34, but does intersect 13, the transformation may be so determined that it will also intersect 24; *e. g.*, by making the plane 2 pass through the intersection of the given line with the plane 4. In this case since u_4 is unaltered we have $\eta_3=1$, $\eta_4=0$, and hence to equate $A_{13}\equiv\eta_3(\eta_1 B_{13}+\eta_2 B_{23})$ to zero we need only to choose $\dfrac{\eta_2}{\eta_1}=-\dfrac{B_{13}}{B_{23}}$. After the transformation we have $A_{13}=A_{24}=0$, and by elimination in (32) we get by aid of (38)

Case II. $\qquad y_1=hy_3, \quad hp(0)y_2=p(1)y_4,$

where

$$h=\frac{A_{23}}{A_{12}}=-\frac{A_{34}}{A_{14}}=-\frac{p(1)A_{12}}{p(0)A_{14}}=\frac{p(1)A_{23}}{p(0)A_{34}}.$$

III. If the line does not intersect any of the three edges 12, 34, 13, ($B_{34}\neq0$, $B_{12}\neq0$, $B_{24}\neq0$) we can determine the transformation so that the face 2 of the tetraedron shall pass through the intersection of the line and the face 3, while the face 4 passes through the intersection of the line and the face 1. Thus, in the new tetraedron, the line intersects the edges 23 and 14, and we have the relations $A_{14}\equiv\eta_3(\eta_1 B_{14}+\eta_2 B_{24})=0$, $A_{23}\equiv\eta_1(\eta_3 B_{23}+\eta_4 B_{24})=0$, which determine the ratios $\dfrac{\eta_1}{\eta_2}$, $\dfrac{\eta_3}{\eta_4}$ desired in this case. The equations of the line can now be written

Case III. $\qquad y_1=lp(1)y_4, \quad lp(0)y_2=-y_3,$

where

$$l=-\frac{A_{42}}{p(1)A_{12}}=\frac{A_{34}}{p(1)A_{13}}=\frac{A_{12}}{p(0)A_{13}}=-\frac{A_{42}}{p(0)A_{34}}.$$

THEOREM XI. *By a change of dependent variable the self-adjoint equation* (25) *with self-adjoint boundary conditions* (26) (27) *may be reduced to an equation* (31) *which is again self-adjoint, and for which the self-adjoint boundary condition may be written in one of three forms characterized respectively by* $A_{12}=A_{34}=0$; $A_{23}=A_{14}=0$; $A_{13}=A_{42}=0$:

Case I. $\sigma y(0)+y_x(0)=0, \quad \tau y(1)+y_x(1)=0,$ σ, τ *constants (including* ∞),
Case II. $y(0)=hy(1), \quad hp(0)y_x(0)=p(1)y_x(1),$ h=*constant,*
Case III. $y(0)=lp(1)y_x(1), \; lp(0)y_x(0)=-y(1),$ l=*constant,*

The number of zeros of the solutions of the equation remains unaltered under this transformation.

The value of the integral $\int_0^1 \frac{\partial \Gamma}{\partial \lambda} u^2 dx$ *is the same as that of the corresponding integral* $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$.

COROLLARY I. *In Case I the transformation may be so chosen that the boundary conditions can be written in one of the special forms*

$$\text{(a)}\ y(0)=y(1)=0; \quad \text{(b)}\ y(0)=y_x(1)=0;$$
$$\text{(c)}\ y_x(0)=y(1)=0; \quad \text{(d)}\ y_x(0)=y_x(1)=0.$$

§ 6. *Oscillation Theorems for the Sturmian Boundary Conditions.*

We propose now to study oscillation theorems for solutions of the differential equation under the first or sturmian case of the general boundary conditions (26) (27). As was shown in the preceding section this case, characterized by the relations $B_{12}=B_{34}=0$ ($A_{12}=A_{34}=0$), may be reduced to a study of the equation

$$(py_x)_x+G(x, \lambda)y=0, \tag{41}$$

under the boundary conditions

$$\sigma y(0)+y_x(0)=0, \quad \tau y(1)+y_x(1)=0, \tag{42}$$

where σ and τ are constants, the important special forms

$$\left.\begin{array}{ll}\text{(a)}\ y(0)=y(1)=0; & \text{(b)}\ y(0)=y_x(1)=0;\\ \text{(c)}\ y_x(0)=y(1)=0; & \text{(d)}\ y_x(0)=y_x(1)=0;\end{array}\right\} \tag{43}$$

being considered as included for the values 0 and ∞. It was further shown that the boundary conditions may always be reduced to one of these special forms.

We proceed now to a study of the nature of the dependence of λ on σ and τ. If $\bar{y}$ and $\bar{\bar{y}}$ are defined as the following fundamental solutions of (41),

$$\bar{y}(0, \lambda)=0, \bar{y}_x(0, \lambda)=1; \quad \bar{\bar{y}}(0, \lambda)=1, \bar{\bar{y}}_x(0, \lambda)=0, \tag{44}$$

any solution $y(x)$ can be written $y=c_1\bar{y}+c_2$. For determination of c_1 and c_2 we have on substitution in (42),

$$\sigma c_2+c_1=0, \quad \tau[c_1\bar{y}(1,\lambda)+c_2\bar{\bar{y}}(1,\lambda)]+c_1\bar{y}_x(1,\lambda)+c_2\bar{\bar{y}}_x(1,\lambda)=0.$$

A necessary and sufficient condition that there be values of c_1, c_2 satisfying the equations is that

$$\begin{vmatrix} 1 & \sigma \\ \tau\bar{y}(1,\lambda)+\bar{y}_x(1,\lambda) & \tau\bar{\bar{y}}(1,\lambda)+\bar{\bar{y}}_x(1,\lambda) \end{vmatrix}=0.$$

Hence each of the parameters σ, τ is a one-valued function of the other and of λ. If we assume that a zero of the solution moves continuously to the right or left, and note what happens as two consecutive zeros pass through the point $x=0$, we see geometrically that for any given σ the condition $\sigma y(0)+y_x(0)=0$ must be satisfied at some stage of the process. It is possible by analysis to ascertain precisely what happens when σ or τ passes over the range $-\infty$, $+\infty$. Corresponding to $\sigma=-\infty$, 0, $+\infty$ one has respectively $y(0)=0$, $y_x(0)=0$, $y(0)=0$, and we shall show that under proper conditions one and only one zero of $y(x)$ is lost or gained by the process. A similar change in τ has a corresponding result.

If in formula (13) we substitute from (42) and the formulae

$$\sigma\frac{\partial y(0)}{\partial\lambda}+\frac{d\sigma}{d\lambda}y(0)+\frac{\partial y_x(0)}{\partial\lambda}=0, \quad \tau\frac{\partial y(1)}{\partial\lambda}+\frac{d\tau}{d\lambda}y(1)+\frac{\partial y_x(1)}{\partial\lambda}=0,$$

obtained from (42) by differentiation with regard to λ, we obtain the relation

$$\int_0^1\frac{\partial G}{\partial\lambda}y^2dx=-p(0)\frac{d\sigma}{d\lambda}y^2(0)+p(1)\frac{d\tau}{d\lambda}y^2(1). \tag{45}$$

Either of the parameters σ, τ may be held fixed. If τ is fixed $\frac{d\tau}{d\lambda}=0$, and in order that λ and σ be one-valued functions of one another over the range $-\infty$, $+\infty$ of σ it is then necessary and sufficient that the integral $\int_0^1\frac{\partial G}{\partial\lambda}y^2dx$ have one sign. From such considerations one may deduce a variety of theorems of which the following are examples.

THEOREM XIII. *If λ'_m, λ'_{m+1} denote two successive characteristic numbers of a solution of (41) for the boundary conditions 43(a), then in order that there be for all σ an intermediate value of λ corresponding to which there is precisely one solution of (41) for the boundary conditions $\sigma y(0)+y_x(0)=0$, $y(1)=0$, it is necessary and sufficient that the integral $\int_0^1\frac{\partial G}{\partial\lambda}y^2dx$ have one sign for all solutions $y(x)$ concerned.*

THEOREM XIV. *If $\bar{\lambda}_m$, $\bar{\lambda}_{m+1}$ denote two successive characteristic numbers of a solution of* (41) *for the boundary conditions*

$$\sigma y(0)+y_x(0)=0, \quad y(1)=0, \quad \sigma=constant,$$

then in order that there be for all τ an intermediate value of λ corresponding to which there is precisely one solution of (41) (42), *it is necessary and sufficient that the integral $\int_0^1 \frac{\partial G}{\partial\lambda} y^2 dx$ have one sign for all solutions $y(x)$ concerned.*

THEOREM XV. *If corresponding to λ', λ'' there are solutions of* (41) (43a) *with n_1 and n_2 zeros respectively, and if $\int_0^1 \frac{\partial G}{\partial\lambda} y^2 dx$ has one sign, then in the interval λ', λ'' there is one and only one value of λ for which* (41) (42) *has a solution $y(x)$ with n zeros $(n_1<n<n_2)$.*

We have seen in § 5 that by a transformation of the dependent variable the equation (41) may be thrown into the form

$$(\bar{p}\bar{y}_x)_x+\bar{G}\bar{y}=0, \quad \int_{\alpha_1}^{\alpha_2} \frac{\partial \bar{G}}{\partial\lambda}\bar{y}^2 dx=\int_{\alpha_1}^{\alpha_2}\frac{\partial G}{\partial\lambda} y^2 dx, \quad \bar{p}(x)>0,$$

where the boundary conditions (42) assume one of the special forms

$\bar{y}(\alpha_1)=\bar{y}(\alpha_2)=0$; $\bar{y}(\alpha_1)=\bar{y}_x(\alpha_2)=0$; $\bar{y}_x(\alpha_1)=\bar{y}(\alpha_2)=0$; $\bar{y}_x(\alpha_1)=\bar{y}_x(\alpha_2)=0$.

As an extension of *Theorems* IV and IVA we have then the following:

THEOREM XVI. *If a zero of one of the functions $\sigma y+y_x$, $\tau y+y_x$ is held fixed, then with increasing λ the zeros of the other move closer to the fixed zero or further away according as $\int \frac{\partial G}{\partial\lambda} y^2 dx$ is positive or negative, the integration extending over the interval between the zeros.*

From the preceding theorems we can deduce still further results.

THEOREM XVII. *If $\int_0^1 \frac{\partial G}{\partial\lambda} y^2 dx$ has one sign, then between any two consecutive λ's corresponding to solutions of* (41) (43a) *or of* (41) (43d) *there will be one each corresponding to solutions of* (41) (43b) *and* (41) (43c), *and between two consecutive λ's for* (41) (43b) *or* (41) (43c) *there will be one each for* (41) (43a) *and* (41) (43d).

COROLLARY: *Denoting by λ_a and λ_d two adjacent characteristic values for solutions of the problem* (41) (43a), (41) (43d) *respectively, and by λ_b, λ_c and λ_b', λ_c' the next greater and next smaller sets for solutions of the problem* (41) (43b), (41) (43c) *respectively, then if the number of zeros of the solution*

corresponding to λ_a is m, the number corresponding to λ_d is $m-1$. When the integral is positive the number of zeros corresponding to λ_b and λ_c is m, and to λ_b' and λ_c' is $m-1$, while if it is negative, the reverse is true.

If we consider the special form* of (41) where $\frac{\partial G}{\partial \lambda} \geqq 0$ and Hypothesis A of § 2 is satisfied while $G(x, -\infty) < 0$ and $= -\infty$ in at least some portion of the interval, we can trace the various values of λ corresponding to successive suites of σ and τ, and state

THEOREM XVIII. *There exists under these hypotheses an infinite set of characteristic numbers $\lambda_1 < \lambda_2 < \lambda_3 \ldots .$, with limit point at $+\infty$ only, for which exist solutions of* (41) (42), *the solution $y_n(x)$ corresponding to $\lambda_n (n=1, 2, 3, \ldots .)$ having $n-1$ zeros within the interval.*

The results for the orthogonal case (§ 1, II) of the special equation

$$(py_x)_x + (q+\lambda k)y = 0, \tag{46}$$

are contained in the foregoing theorem. In the other cases it is preferable to investigate the zeros through a discussion of those of the equation obtained by the transformation used in proving *Theorem* XVI. We have

$$(\bar{p}\bar{y}_x)_x + (\bar{q}+\lambda\bar{k})\bar{y} = 0, \quad \bar{p}(x) > 0, \quad \bar{y}(0)\bar{y}_x(0) = \bar{y}(1)\bar{y}_x(1) = 0,$$

$$\int_0^1 (\bar{p}\bar{y}_x^2 - \bar{q}\bar{y}^2)\,dx = \int_0^1 \bar{k}\bar{y}^2 dx = \int_0^1 ky^2 dx.$$

If the new equation is of the polar form we see at once that there are precisely two solutions of (46) (42) *with n zeros in the interval. If the equation is of the non-definite form there are two integers n_1, n_2, such that for $n < n_1$ there are no solutions of* (46) (42), *for $n > n_1$ there are at least two, while for $n > n_2$ there are precisely two (Theorems VII, VII A).*

§ 7. *Oscillation Theorems for Case II of the Boundary Conditions.*

We shall next consider the exceptional case $B_{42} \equiv \alpha_4\beta_2 - \alpha_2\beta_4 = 0$ for which, as has been seen in § 5, the boundary conditions may be written

$$y(0) = hy(1), \quad hp(0)y_x(0) = p(1)y_x(1), \quad h = \text{constant} \neq 0. \tag{47}$$

Defining the particular solutions $\bar{y}(x, \lambda)$, $\bar{\bar{y}}(x, \lambda)$ by (44), and substituting the solution $y = c_1\bar{y} + c_2\bar{\bar{y}}$ in (47), we obtain equations for c_1 and c_2,

$$c_2 = h[c_1\bar{y}(1, \lambda) + c_2\bar{\bar{y}}(1, \lambda)], \quad hp(0)c_1 = p(1)[c_1\bar{y}_x(1, \lambda) + c_2\bar{\bar{y}}_x(1, \lambda)]. \tag{48}$$

A necessary condition for a solution is the equation

$$D \equiv \begin{vmatrix} h\bar{y}(1, \lambda) & h\bar{\bar{y}}(1, \lambda) - 1 \\ p(1)\bar{y}_x(1, \lambda) - hp(0) & p(1)\bar{\bar{y}}_x(1, \lambda) \end{vmatrix} = 0. \tag{49}$$

* This is a slightly more general condition than that imposed by Birkhoff. *Loc. cit.*

Since for the two solutions we have the well-known formula

$$p(x)\,[\bar{\bar{y}}_x(x,\lambda)\bar{y}(x,\lambda)-\bar{y}_x(x,\lambda)\bar{\bar{y}}(x,\lambda)]=\text{constant}=-p(0), \tag{50}$$

this condition reduces to

$$h^2p(0)\bar{\bar{y}}(1,\lambda)-2hp(0)+p(1)\bar{y}_x(1,\lambda)=0,$$

from which we get by again using (50)

$$h=\frac{1}{\bar{\bar{y}}(1,\lambda)}\pm\frac{\sqrt{p(1)}\sqrt{-\bar{\bar{y}}_x(1,\lambda)\bar{y}(1,\lambda)}}{\sqrt{p(0)}\bar{\bar{y}}(1,\lambda)}. \tag{51}$$

So long as $\bar{\bar{y}}_x(1,\lambda)$ and $\bar{y}(1,\lambda)$ have opposite signs there will be two values of h for each value of λ; when they have the same sign there will be none. We may distinguish three critical cases for which the solutions pass from real to complex:

(α) If $\bar{y}(1,\lambda)=0$, $\bar{\bar{y}}_x(1,\lambda)\neq 0$, the equation (49) can by means of (50) be written $[h\bar{\bar{y}}(1,\lambda)-1]^2p(1)\bar{y}_x(1,\lambda)=0$, and since $\bar{y}$ and $\bar{y}_x$ can not vanish together, it follows that $h\bar{\bar{y}}(1,\lambda)-1=0$. Hence every element of the determinant D vanishes except that in the lower right-hand corner; it follows then that $c_2=0$. The solution $y(x)\equiv c_1\bar{y}(1,\lambda)$ vanishes at $x=0$, and since $h\neq 0$, we have from (47) that it vanishes also at $x=1$. The function $y(x)$ is therefore a solution of (41) (43a).

(β) If $\bar{y}(1,\lambda)\neq 0$, $\bar{\bar{y}}_x(1,\lambda)=0$, we have in the same way $c_1=0$ and the solution $y\equiv c_2\bar{\bar{y}}$ also a solution of (41) (43d).

(γ) If $\bar{y}(1,\lambda)=0$, $\bar{\bar{y}}_x(1,\lambda)=0$ it follows immediately by the same reasoning that all elements of D vanish. Hence c_1 and c_2 may take on any values and λ is a double characteristic number.

On multiplying equation (41) by y and integrating under the boundary conditions (47) we obtain the relation

$$\int_0^1 py_x^2dx=\int_0^1 Gy^2dx. \tag{52}$$

By substitution in (13) from formulae (47) and the further formulae,

$$h\frac{\partial y(1)}{\partial\lambda}=\frac{\partial y(0)}{\partial\lambda}-\frac{dh}{d\lambda}y(1),\quad \frac{dh}{d\lambda}p(0)y_x(0)+hp(0)\frac{\partial y_x(0)}{\partial\lambda}=p(1)\frac{\partial y_x(1)}{\partial\lambda}, \tag{53}$$

obtained from (47) by differentiation with regard to λ, we obtain the fundamental formula

$$\frac{dh}{d\lambda}=-\frac{h\int_0^1\frac{\partial G}{\partial\lambda}y^2dx}{2p(0)y(0)y_x(0)}=-\frac{h^2\int_0^1\frac{\partial G}{\partial\lambda}y^2dx}{2p(1)y(0)y_x(1)}=-\frac{\int_0^1\frac{\partial G}{\partial\lambda}y^2dx}{2p(0)y(1)y_x(0)}. \tag{54}$$

Let us denote by λ_b a characteristic number of our problem for $h=0$; this is a solution of the sturmian case (41) (43b). Near it, as we know from *Theorem* XVII and its corollary, there is another characteristic number λ_c corresponding to $h=\infty$ and to (41) (43c), the two solutions for λ_b and λ_c having the same number of zeros. The parameter value λ_b may be equal to, greater than, or less than λ_c. Of the aggregate of characteristic numbers for the two problems (41) (43a), (41) (43d) let us denote by λ_{ad} the greatest of those smaller than λ_b (or λ_c) and by λ'_{ad} the smallest of those greater. Within the interval λ_{ad}, λ'_{ad} it follows from *Theorem* XVII that there is but one zero and one infinity of h; in other words, but one solution for each of the problems (41) (43b), (41) (43c). On the hypothesis that $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ has one sign we are now in a position to prove that *in this interval of* λ *the function* $h(\lambda)$ *defined by* (51) *is monotone on both of its branches* (the one including $h=0$ and the other $h=\infty$).

To prove this let us in the first place note that for $h=0$ or $h=\infty$ we have a sturmian case for which existence theorems have already been established, and then show that as h passes through either of these values, $\frac{dh}{d\lambda}$ does not change sign. We see at once from (47) that $y_x(1, \lambda(0))=0$; hence $y(1, \lambda(0))\neq 0$, and it follows from (47) that $y(0, \lambda(h))$ changes sign with h. A reference to the first part of (54) shows that $\frac{dh}{d\lambda}$ will then retain its sign. In the same way $y(1, \lambda(\infty))=0$, $y_x(1, \lambda(\infty))\neq 0$; $y_x(0, \lambda(h))$ changes sign as h goes through infinity and $\frac{dh}{d\lambda}$ retains its sign. This establishes the result since it follows from (54) that while $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ retains one sign the only possibility of $\frac{dh}{d\lambda}$ changing sign is when h or $y(0)$ or $y_x(0)$ changes sign.

By definition $y(0, \lambda(0))=0$, $y_x(1, \lambda(0))=0$, and without loss of generality it may be assumed that $y_x(0, \lambda(0))>0$. The discussion then divides itself into two parts according to

Hypothesis I. $y(1, \lambda(0))>0$. Hypothesis II. $y(1, \lambda(0))<0$.

Under the first hypothesis the number of zeros of y (including that at $x=0$) is even, while under the other it is odd. Roughly speaking, we shall see that a gain or loss of a zero comes when h goes through 0 or ∞. Concerning the function $\lambda(h)$ there is now sufficient data to sketch the graph. *We shall first discuss the problem for the assumption* $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx>0$ *and Hypothesis I.*

It has been proved above that $\frac{dh}{d\lambda}$ can not change sign on either branch of the function. On the branch containing $h=0$, $\lambda=\lambda_b$ we see from (54) that $\frac{dh}{d\lambda}<0$, since for that particular point $y(1)>0$, $y_x(0)>0$. For $h=\infty$, $\lambda=\lambda_o$, the number of zeros is the same as that for $h=0$ (*Corollary, Theorem* XVII) and hence under *Hypothesis* I, $y_x(1)$ and $y(0)$ have opposite signs. From the second part of (54) we see then that $\frac{dh}{d\lambda}>0$ on this other branch. Since at the ends of the interval h is not 0, we know that* in cases (α) and (β) either $y(0)$ or $y_x(0)$ will vanish according as λ_{ad}, λ'_{ad} belong to the problem (41) (43a) or (41) (43d); it follows then from (54) that $\frac{dh}{d\lambda}=\infty$.

Under *Hypothesis* II it is readily shown by the same processes that in an interval which we shall call $\bar{\lambda}_{ad}$, $\bar{\lambda}'_{ad}$ to distinguish it from the other, the branch of the function $\lambda(h)$, which contains $h=0$, is monotone increasing and the branch containing $h=\infty$ is monotone decreasing. As λ increases this form of curve will always alternate with that obtained under *Hypothesis* I. When $\int_0^1 \frac{\partial G}{\partial\lambda} y^2 dx<0$, the second form of curve occurs under *Hypothesis* I, and the first under *Hypothesis* II.

Since by definition the intervals can not overlap, the curves can not overlap. If the upper bound of one interval is the lower bound of the next, we have a double point for λ [cf. (γ) above], and one branch of one curve unites with one of the other to form a function monotone throughout. This is what takes place, for example, in the case of the solutions of the equation† $y_{xx}+\lambda y=0$.

---

* In case (γ), $y(0)$ and $y_x(0)$ may be chosen arbitrarily; cf. next succeeding foot-note.

† For this special equation $\bar{y}\equiv\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}x$, $\bar{\bar{y}}\equiv\cos\sqrt{\lambda}x$, and equation (49) becomes

$$2h-(h^2+1)\cos\sqrt{\lambda}=0,$$

from which we obtain the formula $h=\sec\sqrt{\lambda}\pm\tan\sqrt{\lambda}$. For every positive λ the function h is double-valued. The interval $\lambda_{ad}, \lambda'_{ad}$ is $(2m)^2\pi^2$, $(2m+1)^2\pi^2$, while at $(2m+1/2)^2\pi^2$, h becomes zero on one branch and infinite on the other. The monotone decreasing branch $h_1\equiv\sec\sqrt{\lambda}-\tan\sqrt{\lambda}$ may be considered as joined at both ends to monotone decreasing branches in the next intervals. These intervals $(2m-1)^2\pi^2$, $(2m)^2\pi^2$ and $(2m+1)^2\pi^2$, $(2m+2)^2\pi^2$ are of the type $\bar{\lambda}_{ad}, \bar{\lambda}'_{ad}$. The function h_1 decreases from $+\infty$ to $-\infty$ in the interval $(2m-1/2)^2\pi^2$, $(2m+3/2)^2\pi^2$. The function $h_2\equiv\sec\sqrt{\lambda}+\tan\sqrt{\lambda}$ increases monotonely from $-\infty$ to $+\infty$ in the interval $(m+1/2)^2\pi^2$, $(m+5/2)^2\pi^2$, the branches of adjacent intervals uniting as in the other case. Each of the curves h_1 and h_2 cuts two of the other set orthogonally, the points of intersection occurring at the end points $m^2\pi^2$ of the interval λ_{ad}, λ'_{ad}.

It is now easy to write down oscillation theorems for this case. We note in the first place that under *Hypothesis* I the number of zeros is increased by unity as h goes through zero, and under *Hypothesis* II it is decreased by unity so that for $h>0$ the number of zeros is always even, and for $h<0$ it is odd. There are then two solutions with an even number of zeros in the one case and two with an odd number in the other.

THEOREM XIX. *If in an interval of λ there exist two integers m_1, m_2 positive or zero such that there are solutions of*

$$(py_x)_x+G(x,\lambda)y=0 \tag{55}$$

and (43*a*) *with m_1 and m_2 zeros respectively within the interval* 0, 1, *then under the hypothesis that $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ has one sign throughout, there are for $h>0$ two solutions of* (55) (47) *when m is even, and none when m is odd* $(m_1 \leq m \leq m_2)$: *for $h<0$ there are two solutions when m is odd aud none when m is even.*

The special case where $\frac{\partial G}{\partial \lambda}>0$ and G becomes negative for $\lambda=-\infty$, and moreover negatively infinite in at least a part of the interval and G becomes positively infinite for $\lambda=+\infty$ in at least a part of the interval, includes the case discussed by Birkhoff and the detailed theorems derived by him hold also here.*

The orthogonal problem (§ 1, II) of the equation

$$(py_x)_x+(q+\lambda k)y=0 \tag{56}$$

is contained in the special case just discussed. In discussing the polar case we note that it follows from the special case of the formula (52),

$$0<\int_0^1 (py_x^2-qy^2)\,dx=\lambda\int_0^1 ky^2 dx, \tag{57}$$

that $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx \equiv \int_0^1 ky^2 dx$ has the same sign as λ. Hence $\lambda=0$ is not included in the range of values (unless one includes the solution $y\equiv 0$).

THEOREM XX. *In the polar case of equation* (56) *there are precisely two solutions satisfying the boundary conditions* (47) *and oscillating n times* $(n=0, 1, 2, \ldots.)$.

In the non-definite case we have from *Theorem* VII that for sufficiently large values of $|\lambda|$ the integral on the left of (57) is positive and hence we can state

* Case II corresponds to III, p. 260 in Birkhoff's article, *loc. cit.*

THEOREM XXI. *For the non-definite case of equation* (56) *there exist two integers* n_1, n_2 $(n_2 \geqq n_1 \geqq 0)$ *such that for* $n < n_1$ *there is no solution of* (56) (47) *with* n *zeros in the interval; for* $n > n_1$ *there are at least two, while for* $n \geqq n_2$ *there are precisely two.*

§ 8. *Oscillation Theorems for Case III of the Boundary Conditions.*

There remains the most general of the three normal forms obtained for the boundary conditions, viz.:

$$y(0) = lp(1) y_x(1), \quad lp(0) y_x(0) = -y(1), \quad l = \text{constant} \neq 0. \tag{58}$$

Since the discussion follows the same lines as that of the preceding section it will be abbreviated. If we define the two particular solutions $\bar{y}(x, \lambda)$, $\bar{\bar{y}}(x, \lambda)$ as in (44), we get in place of formulae (49), (51),

$$\left.\begin{aligned} D' &\equiv \begin{vmatrix} lp(1)\bar{y}_x(1,\lambda) & lp(1)\bar{\bar{y}}_x(1,\lambda) - 1 \\ \bar{y}(1,\lambda) + lp(0) & \bar{\bar{y}}(1,\lambda) \end{vmatrix} = 0, \\ l &= \frac{1}{p(1)\bar{\bar{y}}_x(1,\lambda)} \pm \frac{\sqrt{\bar{y}_x(1,\lambda)\bar{\bar{y}}(1,\lambda)}}{\sqrt{p(0)\,p(1)\bar{\bar{y}}_x(1,\lambda)}}. \end{aligned}\right\} \tag{59}$$

The critical values are when $\bar{y}_x(1, \lambda) = 0$ and $\bar{\bar{y}}(1, \lambda) = 0$ and the cases may be classified as before.

(α) If $\bar{\bar{y}}(1, \lambda) = 0$, $\bar{y}_x(1, \lambda) \neq 0$, equation (59) can be written

$$\bar{y}(1, \lambda)\,[lp(1)\bar{\bar{y}}_x(1, \lambda) - 1]^2 = 0 \quad \text{or} \quad lp(1)\bar{\bar{y}}_x(1, \lambda) = 1.$$

It follows that $c_1 = 0$ and the solution $y(x, \lambda) \equiv c_2\bar{\bar{y}}(x, \lambda)$ is a solution of the equation with the sturmian boundary condition (43c).

(β) If $\bar{\bar{y}}(1, \lambda) \neq 0$, $\bar{y}_x(1, \lambda) = 0$ we have $c_2 = 0$ and $y \equiv c_1\bar{y}$, which is a solution of (41) (43b).

(γ) If $\bar{\bar{y}}(1, \lambda) = 0$, $\bar{y}_x(1, \lambda) = 0$, it follows as before that D' vanishes identically and λ is a double parameter value.

To replace formula (53) and (54) we have

$$\frac{\partial y(0)}{\partial \lambda} = \frac{dl}{d\lambda} p(1) y_x(1) + lp(1)\frac{\partial y_x(1)}{\partial \lambda},$$

$$lp(0)\frac{\partial y_x(0)}{\partial \lambda} + \frac{dl}{d\lambda} p(0) y_x(0) = -\frac{\partial y(1)}{\partial \lambda},$$

$$\frac{dl}{d\lambda} = -\frac{l\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx}{2p(0)y_x(0)y(0)} = \frac{l^2\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx}{2y(1)y(0)} = \frac{-\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx}{2p(0)y_x(0)p(1)y_x(1)}. \tag{60}$$

It is readily shown as in § 7 that $\frac{dl}{d\lambda}$ does not change sign as l goes through the values 0 or ∞. Let us denote by λ_a, λ_d two adjacent characteristic numbers for the cases $l=0$ and $l=\infty$, respectively (in other words for (41) (43a), (41) (43d)), by λ_{bc} the greatest of the aggregate of characteristic numbers for (41) (43b) and (41) (43c) which are smaller than λ_a, and by λ'_{bc} the smallest of the aggregate larger. It is readily shown as in the previous section that within the interval λ_{bc}, λ'_{bc}, l is a monotone function on the branch containing $l=0$ and on the branch containing $h=\infty$. Let us consider first the case $l=0$; then $y(0, \lambda(0))=0$, $y(1, \lambda(0))=0$ and we can assume $y_x(0, \lambda(0))>0$. There will be two cases to distinguish according as we make

Hypothesis I. $y_x(1, \lambda(0))>0$; Hypothesis II. $y_x(1, \lambda(0))<0$.

Under *Hypothesis* I the number of zeros is always odd, and when l goes through zero from negative to positive it may be seen from (58) that two zeros of the solution $y(x)$ are lost. Under *Hypothesis* II the number of zeros is even, and, as l increases through zero, two zeros of $y(x)$ are gained. It follows from the last part of (60) that the integral $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ and $\frac{dl(0)}{d\lambda}$ have opposite signs. This fixes the sign of $\frac{dl}{d\lambda}$ on one branch. To fix the sign of $\frac{dl}{d\lambda}$ on the other branch let us consider $\lambda(\infty)\equiv\lambda_d$. Under *Hypothesis* I we can argue from the corollary to *Theorem* XVII that $y(0, \lambda(\infty))$ and $y(1, \lambda(\infty))$ have the same sign, and from the third part of (60) that when $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx>0$, $\frac{dl(\infty)}{d\lambda}$ is positive. Under *Hypothesis* II we can easily prove that the situation is reversed, $\frac{dl}{d\lambda}$ being positive on the branch through $h=0$, and negative on the other. When $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx<0$ the two types of curves are interchanged. With increasing λ we see then that in both cases curves of much the same form as the two varieties in the preceding section alternate with one another. If adjacent intervals have the same end-point, a branch from the one will unite with a branch of the other.*

From these data various theorems may be deduced, of which the following is typical.

* For the special equation $y_{xx}+\lambda y=0$ one can set up the formula for $l(\lambda)$ as in foot-note, p. 312. It is found that $l=-\csc\sqrt{\lambda}\pm\cot\sqrt{\lambda}/\sqrt{\lambda}$ and that these monotone functions extend from $-\infty$ to $+\infty$, each of the one set cutting two of the other set in double points $\lambda=(n+1/2)^2\pi^2$ of $l(\lambda)$.

THEOREM XXII. *If in an interval of* λ *there exist two integers* $\mu_1 < \mu_2$, *positive or zero, such that there are solutions of* (55) (43*a*) *with* μ_1 *and* μ_2 *zeros respectively within the interval* 0, 1, *then*

(1) *Under the hypothesis that* $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ *be positive, there are, when* l *is positive, two solutions of* (55) (58) *which have* $2m$ *or* $2m-1$ *zeros in the interval* ($\mu_1 < 2m-1 < 2m < \mu_2$) *and there are, when* l *is negative, two solutions which have* $2m$ *or* $2m+1$ *zeros* ($\mu_1 < 2m < 2m+1 < \mu_2$).

(2) *Under the hypothesis that* $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ *be negative, there are, when* l *is positive, two solutions of* (55) (58) *which have* $2m$ *or* $2m+1$ *zeros in the interval* ($\mu_1 < 2m < 2m+1 < \mu_2$) *and there are, when* l *is negative, two solutions which have* $2m$ *or* $2m-1$ *zeros* ($\mu_1 < 2m-1 < 2m < \mu_2$).

One can make the results of this theorem more specific by giving the conditions necessary to characterize the branch of the function $l(\lambda)$ which is involved. Birkhoff has done this for the special case treated by him (*loc. cit.*, p. 269, I, II), but we shall content ourselves with stating that the same classification may be made in the general case treated here.

Theorems for the special equation (56) analogous to *Theorems* XX, XXI, can be at once written down.

BROWN UNIVERSITY, *January*, 1918.

Theta Modular Groups Determined by Point Sets.

BY ARTHUR B. COBLE.*

Introduction.

In a series of articles already published† by the writer, the study of the properties of a set P_n^k of n discrete points in S_k has been initiated. In these articles particular attention has been paid to certain discontinuous groups defined by the P_n^k, to the invariants of these groups, and to applications to the theory of equations. During the course of the work there appeared, in a special case, a connection between the point set and theta modular functions. It is the purpose of this article to show that this connection must exist in the general case.

An especially interesting type of point set is the self-associated P_{2p+2}^p.‡ This set of $2p+2$ points in S_p has the characteristic geometric property that all the quadrics on $2p+1$ points of the set pass through the remaining point. It has the characteristic algebraic property that complementary determinants formed from the matrix of the coordinates of the points differ by a fixed factor of proportionality. If $p+2$ of the points be chosen as a base in S_p, the remaining p points lie on an S_{p-1}, α, in S_p, and quadrics on the base cut α in quadric sections, all of which are apolar to a quadric Q_α in α. The p points on α are any self-polar p-edron of Q_α. After choosing the base in S_p, p absolute constants are required to fix α, and $\frac{1}{2}p(p-1)$ absolute constants are required to fix the self-polar p-edron of Q_α, whence the number of absolute constants in the self-associated P_{2p+2}^p is $\frac{1}{2}p(p+1)$.

Thus the number of absolute constants of this self-associated point set and the number of moduli of the general theta function of p variables are the same. This might be dismissed as a mere coincidence were it not for the fact that the point set and the moduli are known to be related in a number of

* This investigation has been carried on under the auspices of the Carnegie Institution of Washington, D. C.

† "Point Sets and Cremona Groups," Parts I, II, III, *Transactions of the American Mathematical Society*, 1915–16–17. These are referred to hereafter as P. S. I, II, III respectively.

‡ P. S. I (12).

special cases. A general class of such cases is the set of $2p+2$ points on the rational norm curve R^p in S_p. Obviously all the quadrics on $2p+1$ of these points will pass through the remaining one, and the set is self-associated. Let us call it a *hyperelliptic* self-associated set. It is well known that such a set defines the hyperelliptic theta functions with $2p-1$ independent moduli. Nor is the relation here existing peculiar to the hyperelliptic case. For when p is 3, the smallest value of p for which the theta functions are not necessarily hyperelliptic, it is well known that the self-associated set of eight points in S_3 will define a quartic curve of genus 3, and thereby also the moduli of the theta functions of genus 3. In all of these cases the properties of the point set and of the modular functions have been developed sufficiently to make the connection quite clear.*

In P. S. I § 6, projectively equivalent ordered sets P_n^k were mapped upon a point P of a space $\Sigma_{k(n-k-2)}$ and by permutation of the points of P_n^k a group $G_{n!}$ in Σ was derived. A conjugate set of points P in Σ under $G_{n!}$ represents all sets P_n^k projective in some order to each other. In P. S. II § 3, the congruence of sets P_n^k was defined. Two point sets P_n^k and $P_n'^k$ are congruent if the one arises from the other by means of a sequence of operations which are either projectivities or those particular Cremona transformations which occur when the variables are inverted. In applying an operation of the latter type it is understood that the $k+1$ F-points of the Cremona transformation belong to both congruent sets P_n^k and $P_n'^k$, and that the remaining $n-(k+1)$ points of each set form corresponding pairs of the Cremona transformation. It then appeared that all point sets congruent in some order to a given set P_n^k were mapped in Σ by points P which formed a conjugate set under a group $G_{n,k}$ in $\Sigma_{k(n-k-2)}$. This infinite, discontinuous Cremona group $G_{n,k}$ is thereby projectively defined by the point set P_n^k. In the main this article is concerned with the structure of this group.

Though generating elements of $G_{n,k}$ have been given (P. S. II(15)), it is more convenient to handle certain isomorphic groups of linear transformations, particularly the group $g_{n,k}$. The operation which transforms a point set P_n^k into a congruent set $P_n'^k$ will transform an algebraic spread in S_k of order x_0, and multiplicities $x_1, \ldots, x_n$ at the points of P_n^k into an algebraic spread of order x_0' with multiplicities $x_1', \ldots, x_n'$ at the points of $P_n'^k$. Then x' is the linear transform of x under an element of $g_{n,k}$. The group $g_{n,k}$ is simply

* For the case $p=3$, however, explicit formulae for the invariants of the point set in terms of the modular functions are still lacking.

isomorphic with $G_{n,k}$ except in a few particular cases with which we are not concerned here. The elements of $g_{n,k}$ have determinants ± 1 and integer coefficients. Hence the group $g_{n,k}$ admits of a modular theory precisely similar to that of the groups of transformations of the periods of the multiply periodic functions. For example, it is clear that

(1) *The elements of $g_{n,k}$ whose coefficients reduce modulo m to those of the identity, form an infinite invariant subgroup of $g_{n,k}$ whose factor group $g^{(m)}_{n,k}$ is finite.*

We shall determine in this article the order and the structure of all the groups $g^{(2)}_{n,k}$. Since $g_{n,k}$ has a quadratic invariant we should expect these groups to be subgroups of the group of a null system in the finite geometry mod. 2.

In § 1 the generators and invariants of $g_{n,k}$ are given, and these are reduced mod. 2, to generators and invariants respectively of $g^{(2)}_{n,k}$. Certain new invariants are derived and the sets of linear forms conjugate under $g^{(2)}_{n,k}$ are tabulated. These linear forms are permuted under $g^{(2)}_{n,k}$ just as certain geometric objects in the finite geometry of the null system are permuted under certain subgroups of the group of the null system. The identification of the forms and the corresponding geometric objects is made in § 3, there being sixteen cases according as

$$k = 4\lambda + \varkappa, \quad n = 4\mu + \nu \quad (\varkappa, \nu = 0, 1, 2, 3). \tag{2}$$

The finite geometry is exhibited in terms of the basis notation. This notation is recapitulated in § 2, and is there used to determine the structure of the groups which appear in § 3.*

In order to impose conveniently the conditions for a self-associated set, we use the *ultra-elliptic* point set P^k_n, *i. e.*, n-points on the normal elliptic curve E^{k+1} in S_k with elliptic parameters $u_1, \ldots, u_n$. Thus in S_p the $2p+2$-points cut out on E^{p+1} by a quadric evidently constitute an ultra-elliptic self-associated set, and the condition for self-association is merely

$$u_1 + u_2 + \ldots + u_{2p+2} = 0. \tag{3}$$

For $p=2$ and $p=3$ every self-associated set is ultra-elliptic, and it can be shown that this is true for $p=4$ also. For further values of p the ultra-elliptic self-associated set must be special since it contains only $2p+2$ absolute constants—one for the E^{p+1} and $2p+1$ for the parameters of all but one of the points.

*The statements are drawn from two earlier papers of the writer: "An Application of Finite Geometry to the Characteristic Theory of the Odd and Even Theta Functions," *Trans. Amer. Math. Soc.*, Vol. XIV (1913), and "An Isomorphism between Theta Characteristics and the $(2p+2)$-point," *Annals of Math.*, Vol. XVII, Ser. 2 (1916). These are referred to respectively as F. G. I and F. G. II.

If an ultra-elliptic set P_n^k on E^{k+1} be transformed into a congruent ultra-elliptic set $P_n'^k$ on E'^{k+1}, the two elliptic curves are projective. If E'_{k+1} be projected upon E^{k+1} the set $P_n'^k$ is projected upon a set on E^{k+1} with parameters $u'_1, \ldots, u'_n$. Then the $u''s$ and $u's$ are linearly related under an element of the group $e_{n,k}$ which is simply isomorphic with $g_{n,k}$ (P. S. II (33)). Thus, though the ultra-elliptic set is special, it can be used to determine the structure of the groups determined by the general set.

In §4 the relation of $g_{n,k}^{(2)}$ to modular groups determined by $e_{n,k}$ is discussed, and the restriction to self-associated sets P_{2p+2}^p is made by the use of (3).

We shall assume throughout that $n \geqq k+3$.

§1. *The Generators, Invariants and Conjugate Linear Forms of $g_{n,k}^{(2)}$.*

The group $g_{n,k}$ is generated (P. S. II §5) by transpositions such as T_{12} of the variables $x_1, \ldots, x_n$ and the involutory element

$$A_{1,2,\ldots,k+1}: \begin{array}{l} x'_i = x_i + [(k-1)x_0 - x_1 - x_2 \ldots - x_{k+1}] \quad (i=0, 1, 2, \ldots, k+1), \\ x'_j = x_j \quad (j=k+1, \ldots, n). \end{array}$$

These generators belong in a conjugate set. The group has an invariant quadric form M, and an invariant point O and linear form L which are pole and polar as to M. These are

$$\begin{aligned} M &= (k-1)x_0^2 - (x_1^2 + x_2^2 + \ldots + x_n^2), \\ L &= (k+1)x_0 - (x_1 + x_2 + \ldots + x_n), \\ O &= k+1, \ k-1, \ k-1, \ \ldots, k-1. \end{aligned}$$

If these generators be reduced mod. 2, and thereafter combined mod. 2, they become the generators of $g_{n,k}^{(2)}$. The above invariants similarly reduced become invariants of $g_{n,k}^{(2)}$.

Thus $g_{n,k}^{(2)}$ is generated by transpositions such as T_{12} and $A_{1,\ldots,k+1}$ where

$$(4) \quad A_{1,\ldots,k+1}: \begin{array}{ll} k \text{ even.} & k \text{ odd.} \\ x'_i = x_i + (x_0 + x_1 + \ldots + x_{k+1}), & x'_i = x_i + (x_1 + \ldots + x_{k+1}), \\ x'_j = x_j, & x'_j = x_j, \end{array}$$
$$(i=0, 1, \ldots, k+1; \ j=k+2, \ldots, n).$$

Writing M above in polarized form before reducing we have as invariants of $g_{n,k}^{(2)}$:

$$(5) \quad \left\{ \begin{array}{ll} k \text{ even.} & k \text{ odd.} \\ M(x,y) = x_0y_0 + x_1y_1 + \ldots + x_ny_n, & M(x,y) = x_1y_1 + \ldots + x_ny_n, \\ L = x_0 + x_1 + \ldots + x_n, & L = x_1 + \ldots + x_n, \\ O = 1, 1, \ldots, 1; & O = 0, 1, \ldots, 1. \end{array} \right.$$

It may happen that $g^{(2)}_{n,k}$ will have other quadratic invariants. Any quadratic form symmetric in $x_1, \ldots, x_n$ after possible subtraction of $M(x,x)$ takes the form

$$(6) \qquad \alpha x_0^2+\beta x_0(x_1+\ldots+x_n)+\gamma(x_1x_2+\ldots+x_{n-1}x_n).$$

For k even let $x_i'=x_i+D$, $x_j'=x_j$ be $A_{1,\ldots,k+1}$. Under this element (6) acquires the increment

$$\alpha D^2+\beta\{Dx_0+D(x_1+\ldots+x_n)+D^2\}+\gamma\left\{\binom{k+1}{2}D^2+(x_{k+2}+\ldots+x_n)D\right\},$$

and this increment must vanish. After factoring out D, we find from the typical terms in x_0, x_1, x_n that $\alpha+\gamma\dfrac{k}{2}\equiv\beta+\gamma\equiv 0$ mod. 2. Hence defining $\varkappa$ as in (2) we have when $\varkappa=0$ that $\alpha=0$, $\beta=\gamma=1$; and when $\varkappa=2$ that $\alpha=\beta=\gamma=1$. When k is odd the increment is

$$\alpha D^2+\beta D(x_1+\ldots+x_n)+\gamma\left\{\binom{k+1}{2}D^2+D^2\right\},$$

whence $\alpha+\beta+\left\{\dfrac{k+1}{2}+1\right\}\gamma\equiv\beta\equiv 0$. If $\varkappa=1$, then $\alpha=\beta=0, \gamma=1$; and if $\varkappa=3$, then $\beta=0$, $\alpha=\gamma=1$. Hence

(7) *The* $g^{(2)}_{n,k}$ *has the additional invariant quadratic form:*

$$\begin{aligned}
\varkappa=0,\ M'&= \phantom{x_0^2+{}} x_0(x_1+\ldots+x_n)+(x_1x_2+\ldots+x_{n-1}x_n);\\
\varkappa=2,\ M'&=x_0^2+x_0(x_1+\ldots+x_n)+(x_1x_2+\ldots+x_{n-1}x_n);\\
\varkappa=1,\ M'&= (x_1x_2+\ldots+x_{n-1}x_n);\\
\varkappa=3,\ M'&=x_0^2 +(x_1x_2+\ldots+x_{n-1}x_n).
\end{aligned}$$

Thus in all cases the $g^{(2)}_{n,k}$ is simply or multiply isomorphic with the group of a quadric or with a subgroup of such a group. In order to determine it more precisely we shall determine its effect upon the system of integer linear forms whose coefficients are reduced mod. 2, *i. e.*, upon the spaces S_{n-1} in the S_n of $x_0, x_1, \ldots, x_n$. Any element of $g^{(2)}_{n,k}$ is completely defined if its effect upon any $n+1$ linearly independent forms is known.

All of these forms are comprised under the following types:

$$(8) \qquad B_{1,2,\ldots,l}=x_1+x_2+\ldots+x_l, \quad C_{1,2\ldots,l}=x_0+x_1+x_2+\ldots+x_l.$$

We shall denote by $B(l)$ and $C(l)$ respectively the aggregate of forms (8) obtained by taking all sets of l variables from $x_1, \ldots, x_n$; and we shall denote further by b_1, b_2, b_3, b_4 and c_0, c_1, c_2, c_3 the aggregate of forms $B(l)$,

$C(l)$ for which $l \equiv 1, 2, 3, 4$ or $l \equiv 0, 1, 2, 3$ mod. 4 respectively. From these aggregates the invariant form

$$\begin{aligned} C_{1,2,\ldots,n} &= x_0 + x_1 + \ldots + x_n \quad (k \text{ even}), \\ B_{1,2,\ldots,n} &= \phantom{x_0 +{}} x_1 + \ldots + x_n \quad (k \text{ odd}) \end{aligned}$$

is to be excluded.

Let us consider first the

CASE I: k even.

It is then quite clear that

(9) *A form* $\frac{B(l)}{C(l)}$ *is unaltered, or is transformed into a form* $\frac{C(l+k+1-2r)}{B(l+k+1-2r)}$, *by* $A_{1,\ldots,k+1}$ *according as* $A_{1,\ldots,k+1}$ *has an* $\frac{\textit{even}}{\textit{odd}}$ *or* $\frac{\textit{odd}}{\textit{even}}$ *number* r *of subscripts in common with* $\frac{B(l)}{C(l)}$. *If the form is altered, the subscripts of the transformed form are those which belong either to* $A_{1,\ldots,k+1}$, *or to the original form, but not to both.*

When a B is transformed into a C, and this C back into a B', or a C into a B and this B back into a C', the number of subscripts is increased by $2k+2-2(r+r')$ where r, r' is odd, even or even, odd. Thus forms $B(l), B(l')$ or forms $C(l)$, $C(l')$ are conjugate only when $l \equiv l'$ mod. 4. Moreover, forms $B(l)$, $C(l')$ are conjugate only when $l' \equiv l+k+1-2r$ mod. 4 (r odd). The latter condition depends upon the value of k and we find that

(10) *Under* $g^{(2)}_{n,k}$, *k even, there are four sets of conjugate linear forms, namely:*

$$\begin{aligned} \varkappa=0:\ & b_1, c_0;\ b_2, c_1;\ b_3, c_2;\ b_4, c_3; \\ \varkappa=2:\ & b_1, c_2;\ b_2, c_3;\ b_3, c_0;\ b_4, c_1; \end{aligned}$$

the linear form L being excluded.

That all of the linear forms indicated actually occur in a conjugate set is readily verified. Indeed, beginning with $B(y)$ $(y=1, 2, 3, 4)$ we obtain $C(y+k-1)$ and also, if $y=3$ or 4, $C(y+k-5)$. From these we get in turn all the admissible forms $B(y), B(y+4), \ldots, B(y+2k)$; etc.

Every linear form is paired by addition to the invariant form L with another linear form. These pairs are permuted by $g^{(2)}_{n,k}$ as entities. For k even the pairs of forms are $B(l), C(n-l)$. Recalling from (2) that $n=4\mu+\nu$ $(\nu=0, 1, 2, 3)$ we have the following table of conjugate forms and their pairs:

$$
(11)\qquad
\left\{
\begin{array}{lllll}
 & \multicolumn{2}{c}{\varkappa=0} & \multicolumn{2}{c}{\varkappa=2} \\
\nu=0\left\{\begin{matrix} \\ \end{matrix}\right. & \begin{matrix} b_1, c_0 \\ c_3, b_4 \end{matrix} & \begin{matrix} b_2, c_1 \\ c_2, b_3 \end{matrix} & \begin{matrix} b_1, c_2 \\ c_3, b_2 \end{matrix} & \begin{matrix} b_3, c_0 \\ c_1, b_4 \end{matrix} \\
\nu=2\left\{\begin{matrix} \\ \end{matrix}\right. & \begin{matrix} b_1, c_0 \\ c_1, b_2 \end{matrix} & \begin{matrix} b_3, c_2 \\ c_3, b_4 \end{matrix} & \begin{matrix} b_1, c_2 \\ c_1, b_4 \end{matrix} & \begin{matrix} b_2, c_3 \\ c_0, b_3 \end{matrix} \\
\nu=1\left\{\begin{matrix} \\ \end{matrix}\right. & \begin{matrix} b_2, c_1 \\ c_3, b_4 \end{matrix} & \left\{\begin{matrix} b_1 \\ c_0 \end{matrix}\right. \left\{\begin{matrix} b_3 \\ c_2 \end{matrix}\right. & \begin{matrix} b_1, c_2 \\ c_0, b_3 \end{matrix} & \left\{\begin{matrix} b_2 \\ c_3 \end{matrix}\right. \left\{\begin{matrix} b_4 \\ c_1 \end{matrix}\right. \\
\nu=3\left\{\begin{matrix} \\ \end{matrix}\right. & \begin{matrix} b_1, c_0 \\ c_2, b_3 \end{matrix} & \left\{\begin{matrix} b_2 \\ c_1 \end{matrix}\right. \left\{\begin{matrix} b_4 \\ c_3 \end{matrix}\right. & \begin{matrix} b_2, c_3 \\ c_1, b_4 \end{matrix} & \left\{\begin{matrix} b_1 \\ c_2 \end{matrix}\right. \left\{\begin{matrix} b_3 \\ c_0 \end{matrix}\right.
\end{array}
\right.
$$

Within any compartment conjugate forms are in line, paired forms in column and forms both conjugate and paired are indicated by $\{$.

Turning next to the

CASE II: k odd,

we find that

(12) *A form* $\dfrac{B(l)}{C(l)}$ *is unaltered, or is transformed into a form* $\dfrac{B(l+k+1-2r)}{C(l+k+1-2r)}$, *by* $A_{1,\ldots,k+1}$ *according as* $A_{1,\ldots,k+1}$ *has an* $\dfrac{even}{odd}$ *or* $\dfrac{odd}{even}$ *number* r *of subscripts in common with* $\dfrac{B(l)}{C(l)}$.

The subscripts of the transformed form are found by the same rule as above. We find in all six sets of conjugate forms and the table analogous to (10) is:

$$
(13)\qquad
\left\{
\begin{array}{l}
\varkappa=1:\ b_1;\ b_2;\ b_3;\ b_4;\ c_0,\ c_2;\ c_1,\ c_3; \\
\varkappa=3:\ b_1,\ b_3;\ b_2,\ b_4;\ c_0;\ c_1;\ c_2;\ c_3.
\end{array}
\right.
$$

where, as before, the form L is excluded.

On adding the form $L=B_{1\,\ldots,n}$ the forms $B(l), B(n-l)$ are paired, and also the forms $C(l), C(n-l)$. Hence the table formed, as in (11), of conjugate and paired forms is as follows:

$$
(14)\qquad
\left\{
\begin{array}{lcc}
 & \varkappa=1 & \varkappa=3 \\
\nu=0\left\{\begin{matrix} \\ \end{matrix}\right. & \begin{matrix} b_1 \\ b_3 \end{matrix}\ \{b_2\ \{b_4\ \{c_0\ \{c_2\ \left\{\begin{matrix} c_1 \\ c_3 \end{matrix}\right. & \left\{\begin{matrix} b_1 \\ b_3 \end{matrix}\right. \{b_2\ \{b_4\ \{c_0\ \{c_2\ \begin{matrix} c_1 \\ c_3 \end{matrix} \\
\nu=2\left\{\begin{matrix} \\ \end{matrix}\right. & \{b_1\ \begin{matrix} b_2 \\ b_4 \end{matrix}\ \{b_3\ \left\{\begin{matrix} c_0 \\ c_2 \end{matrix}\right. \left\{\begin{matrix} c_1 \\ c_3 \end{matrix}\right. & \left\{\begin{matrix} b_1 \\ b_3 \end{matrix}\right. \left\{\begin{matrix} b_2 \\ b_4 \end{matrix}\right. \begin{matrix} c_0 \\ c_2 \end{matrix}\ \{c_1\ \{c_3 \\
\nu=1\left\{\begin{matrix} \\ \end{matrix}\right. & \begin{matrix} b_1 & b_2 & c_0, c_2 \\ b_4 & b_3 & c_1, c_3 \end{matrix} & \begin{matrix} b_1, b_3 & c_0 & c_2 \\ b_2, b_4 & c_1 & c_3 \end{matrix} \\
\nu=3\left\{\begin{matrix} \\ \end{matrix}\right. & \begin{matrix} b_1 & b_3 & c_0, c_2 \\ b_2 & b_4 & c_3, c_1 \end{matrix} & \begin{matrix} b_1, b_3 & c_0 & c_2 \\ b_4, b_2 & c_3 & c_1 \end{matrix}
\end{array}
\right.
$$

In order to identify these linear forms with certain configurations of points and O, E quadrics associated [cf. F. G. I, §§ 1, 3] with a linear complex or null system C_p in an odd space S_{2p-1}, the basis notation described in the next section is very convenient.

§ 2. *The Basis Notation for C_p. Certain Subgroups of the Group GC_p.*

1°. With $2p+2$ subscripts $1, 2, \ldots, 2p+2$, the $2^{2p}-1$ points of S_{2p-1} (mod. 2) are indicated by

$$P_{12},\ P_{1234},\ P_{123456},\ \ldots,\quad \text{where}\quad P_{1,2,\ldots,2k}=P_{2k+1,\ldots,2p+2}.$$

Furthermore, $P_{1,2,\ldots,2p+2}$ represents no point, and among the subscripts of a point like subscripts cancel. The three points

$$P_{i_1i_2\ldots i_{2k}},\quad P_{j_1j_2\ldots j_{2l}},\quad P_{i_1i_2\ldots i_{2k}j_1j_2\ldots j_{2l}}$$

are on a line. By means of this condition all linear spaces in S_{2p-1} can be constructed [F. G. II, § 1].

2°. The points $P_{i_1i_2\ldots}$ and $P_{j_1j_2\ldots}$ are syzygetic or azygetic, according as they have an even or an odd number of common subscripts. This relation between the two points is mutual and a point is syzygetic to itself. The locus of points syzygetic to a given point is an S_{2p-2} on the given point, and is the null space of the given point in the null system C_p. Each null space may be named like its null point [F. G. II, Theo. 1].

3°. The 2^{2p} quadrics Q, whose polar system coincides with the null system C_p, divide into $2^{p-1}(2^p+1)E$ quadrics each containing $2^{p-1}(2^p+1)-1$ points and $2^{p-1}(2^p-1)O$ quadrics each containing $2^{p-1}(2^p-1)-1$ points [F. G. I, § 3].

4°. In the base notation the E quadrics are denoted by

$$Q_{1,2,\ldots,p+1-4j}=Q_{p+1-4j+1,\ldots,2p+2}\qquad (j=0, 1, \ldots);$$

and the O quadrics are denoted by

$$Q_{1,2,\ldots,p-1-4j}=Q_{p-1-4j+1,\ldots,2p+2}\qquad (j=0, 1, \ldots).$$

Thus if p is odd there is a quadric Q without subscripts, and this is an O or E quadric according as $p\equiv 1$ or 3, mod. 4. A pair of quadrics determines a point, and a quadric and a point determines another quadric, by virtue of the relation

$$Q_{i_1i_2\ldots}+Q_{j_1j_2\ldots}+P_{i_1i_2\ldots j_1j_2\ldots}=0\quad \text{[F. G. II, Theos. 4, 6].}$$

5°. A point lies on a quadric if half the number of subscripts in a symbol for the point together with the number of subscripts common to the symbols of the point and quadric is even [F. G. II, Theo. 7].

6°. The group GC_p of the null system has the order

$$NC_p = 2^{p^2}(2^{2p}-1)(2^{2p-2}-1)\ldots(2^2-1).$$

It is generated by a conjugate set of involutions $I_{i_1 i_2 \ldots}$ each of which is associated with a $P_{i_1 i_2 \ldots}$. The $I_{i_1 i_2 \ldots}$ leaves every point syzygetic with $P_{i_1 i_2 \ldots}$ unaltered, and sends every point P azygetic with $P_{i_1 i_2 \ldots}$ into the point $P+P_{i_1 i_2 \ldots}$ [F. G. I, § 1].

7°. The group of an O or E quadric has the order

$$NE_p = 2^{p^2-p+1}(2^p-1)(2^{2p-2}-1)(2^{2p-4}-1)\ldots(2^2-1);$$
$$NO_p = 2^{p^2-p+1}(2^p+1)(2^{2p-2}-1)(2^{2p-4}-1)\ldots(2^2-1).$$

The involution $I_{i_1 i_2 \ldots}$ transforms a quadric Q into itself if $P_{i_1 i_2 \ldots}$ is *not* on Q, otherwise it transforms Q into $Q+P_{i_1 i_2 \ldots}$. The group of the quadric is generated by the involutions I attached to the points not on the quadric. This group is simply transitive on the set of points on the quadric and on the set of points not on the quadric. The points on Q added to Q furnish the remaining quadrics of the same type as Q; the points not on Q furnish the quadrics of the opposite type [F. G. I, §3, II, Theo. 7].

8°. The process of section and projection by a null space and from a null point is described in F. G. I, § 2, and used in § 5. So far as we shall need it here the process is as follows: To fix ideas let the point be P_{12}, its null space be L_{12}. Then the points on L_{12} apart from P_{12} are paired on null lines of the form

$$P_{12}, \quad P_{i_1 i_2 \ldots i_{2j}}, \quad P_{12 i_1 i_2 \ldots i_{2j}}.$$

Projected from P_{12} these lines become the $2^{2\pi}-1$ points of an $S_{2\pi-1}$ $(\pi = p-1)$ whose basis notation has subscripts 3, 4, ..., $2p+2$, and the above line corresponds to the point $P'_{i_1 i_2 \ldots i_{2j}}$ in $S_{2\pi-1}$. The remaining $2^{2\pi}$ ordinary lines on P_{12} of the form

$$P_{12}, \quad P_{1 i_1 i_2 \ldots i_{2j-1}}, \quad P_{2 i_1 i_2 \ldots i_{2j-1}}$$

have no trace in $S_{2\pi-1}$. If points $P_{i_1 i_2 \ldots}$, $P_{j_1 j_2 \ldots}$ on L_{12} give rise to null lines on P_{12}, or points $P'_{i_1 i_2 \ldots}$, $P'_{j_1 j_2 \ldots}$ in $S_{2\pi-1}$, these points P' are syzygetic or azygetic according as the original points P are syzygetic or azygetic. Thus the null system C_p in S_{2p-1} determines its projection and section C'_π in $S_{2\pi-1}$.

9°. The 2^{2p-1} quadrics on P_{12} are

$$Q_{1 i_1 i_2 \ldots} = Q_{2 j_1 j_2 \ldots},$$

the sets of subscripts being complementary. These quadrics are paired under I_{12} into pairs $Q_{1 i_1 i_2 \ldots}$, $Q_{2 i_1 i_2 \ldots}$. The members of a pair have the same section

by L_{12} and the null lines on P_{12} are either generators of or tangents to both quadrics. The points of $S_{2\pi-1}$ corresponding to a common generator or a common tangent of the pair is a point on or off respectively the quadric $Q'_{i_1i_2\ldots\ldots}$ in $S_{2\pi-1}$. In this way the 2^{2p-1} quadrics on P_{12} give rise to $2^{2\pi}$ pairs or to the $2^{2\pi}$ quadric Q' in $S_{2\pi-1}$ associated with C'_π. The quadric Q' is an E or O quadric according as the original pair is a pair of E or a pair of O quadrics.

We shall now apply the above notation to derive the generators, the order, and the structure of certain subgroups of GC_p. The first group which we shall consider is that group H_{12} which leaves the point P_{12} unaltered. There being $2^{2p}-1$ points P_{12} the order of H_{12} is

$$NH_{12}=NC_p\div(2^{2p}-1)=2^{2p-1}NC_\pi \quad (\pi=p-1).$$

The null lines on P_{12} are permuted among themselves by H_{12} according to a group GC'_π of order NC_π. For if $P_{i_1i_2\ldots\ldots}$ or $P_{12i_1i_2\ldots\ldots}$ is any point on L_{12}, then either $I_{i_1i_2\ldots\ldots}$ or $I_{12i_1i_2\ldots\ldots}$ will permute the null lines on P_{12} just as the involution $I'_{i_1i_2\ldots\ldots}$ attached to the point $P'_{i_1i_2\ldots\ldots}$ of the derived space $S_{2\pi-1}$ will permute the points of this space. But these involutions generate the group GC'_π. On the other hand the elements of H_{12} must leave the derived null system C'_π unaltered. Hence, to account for the above order of H_{12}, we must determine the elements of H_{12}, which leave P_{12} and every null line on it unaltered.

We had noted that $I_{i_1i_2\ldots\ldots}$ and $I_{12i_1i_2\ldots\ldots}$ effect the same involution on the null lines on P_{12}. Since two involutions I are permutable if their points are syzygetic, we see that the involution

$$T_{i_1i_2\ldots\ldots}=I_{i_1i_2\ldots\ldots}I_{12i_1i_2\ldots\ldots}=I_{12i_1i_2\ldots\ldots}I_{i_1i_2\ldots\ldots}$$

effects the identical permutation of the null lines. The same is true of the involution

$$S_{i_1i_2\ldots\ldots}=I_{12}T_{i_1i_2\ldots\ldots}=T_{i_1i_2\ldots\ldots}I_{12}=I_{12}I_{i_1i_2\ldots\ldots}I_{12i_1i_2\ldots\ldots}.$$

The two involutions $S_{i_1i_2\ldots\ldots}$, $T_{i_1i_2\ldots\ldots}$ are associated with a null line on P_{12}, the former symmetrically and the latter with P_{12} isolated, and may therefore be named by the point $P'_{i_1i_2\ldots\ldots}$ of $S_{2\pi-1}$. We shall therefore speak of $S_{P'}$ and $T_{P'}$ where P' is any one of the $2^{2\pi}-1$ points of $S_{2\pi-1}$. The $2(2^{2\pi}-1)$ elements $T_{P'}$, $S_{P'}$ and the two elements 1, I_{12} constitute the group $\overline{H}_{12}$, which leaves P_{12} and every null line on it unaltered.

The group $\overline{H}_{12}$ is Abelian with involutory elements and its multiplication table is:

$$I_{12}S_{P'}=T_{P'}, \quad I_{12}T_{P'}=S_{P'};$$

if P', $\bar{P}'$ are syzygetic, then

$$T_{P'}T_{\bar{P}'}=S_{P'+\bar{P}'}, \quad T_{P'}S_{\bar{P}'}=T_{P'+\bar{P}'}, \quad S_{P'}S_{\bar{P}'}=S_{P'+\bar{P}'};$$

if P', $\bar{P}'$ are azygetic, then

$$T_{P'}T_{\bar{P}'}=T_{P'+\bar{P}'}, \quad T_{P'}S_{\bar{P}'}=S_{P'+\bar{P}'}, \quad S_{P'}S_{\bar{P}'}=T_{P'+\bar{P}'}.$$

The first two of these relations follow directly from the definitions of S, T; the first in each of the next sets is deduced by verifying the effect of the two members upon the $2p-1$ linearly independent points $P_{12}, P_{13}, \ldots, P_{1,2p+2}$; the remaining ones follow directly from these by inserting factors I_{12}.

The elements $T_{P'}$, $S_{P'}$ of $\bar{H}_{12}$ will interchange or leave unaltered the points on a null line of P_{12} which corresponds to $\bar{P}'$ in $S_{2\pi-1}$ according as P' and $\bar{P}'$ are azygetic or syzygetic. We ask for elements of $\bar{H}_{12}$ which effect the identical collineation in L_{12}. Such an element would leave unaltered all the S_{2p-2}'s on P_{12}; since these are null spaces of points on L_{12}, and therefore *all* the lines on P_{12}. If then it sends the point $P_{1i_1i_2\ldots}$ outside of L_{12} into the point $P_{2i_1i_2\ldots}$ it must interchange the null spaces of these points, and therefore interchange the points in which any ordinary line on P_{12} meets these spaces. Thus 1, I_{12} are the only elements of $\bar{H}_{12}$ which leave every point on L_{12} unaltered.

The group H_{12} contains subgroups simply isomorphic with the GC'_{π} and in fact a conjugate set of $2^{2\pi}$ such subgroups. Let $P_{1i_1i_2\ldots}$, $P_{2i_1i_2\ldots}$ be the pair of points on any one of the $2^{2\pi}$ ordinary lines on P_{12}. Their null spaces cut L_{12} in the same $S_{2\pi-1}$ which meets every null line on P_{12} in a single point $\bar{P}$. The $2^{2\pi}-1$ involutions $I_{\bar{P}}$ generate a group isomorphic with the group GC'_{π} on the null lines. Moreover, the group thus generated can contain no element of $\bar{H}_{12}$ since the order of the group of P_{12} and $P_{1i_1i_2\ldots}$ is $NH_{12}\div 2^{2p-1}=NC'_{\pi}$. A group of this kind occurs for each one of the conjugate set of $2^{2\pi}$ ordinary lines. Incidentally we have shown that

(15) *The group which leaves two azygetic points in S_{2p-1} each unaltered, has the order NC_{π}, and is generated by the involutions I attached to all points syzygetic with the two given points.*

No element $T_{P'}$ or $S_{P'}$ can leave an ordinary line on P_{12} unaltered. For if it did the one or the other of the two would leave each of the two remaining points of the line unaltered, and the group of one of these points and P_{12} would have an element in common with $\bar{H}_{12}$ contrary to what has been proved above. Hence

(16) *The subgroup H_{12} of GC_p, which leaves a point P_{12} and its null space L_{12} unaltered has the order $2 \cdot 2^{2\pi} \cdot NC'_\pi$ ($\pi = p-1$) and is generated by the involutions I attached to points syzygetic with P_{12}. It has an invariant subgroup $\bar{H}_{12}$ of order $2 \cdot 2^{2\pi}$ which leaves every null line on P_{12} unaltered. The factor group of $\bar{H}_{12}$ under H_{12} is a GC_π. Also $\bar{H}_{12}$, an Abelian group, has an invariant g_2 which leaves every point on L_{12} unaltered. The factor group of g_2 under $\bar{H}_{12}$ is a regular group on the $2^{2\pi}$ ordinary lines on P_{12}. The group H_{12} has a set of $2^{2\pi}$ conjugate subgroups GC_π and is the direct product of $\bar{H}_{12}$ and any one of these subgroups.*

Let Q be a quadric on P_{12}. Its section by L_{12} and projection from P_{12} is a quadric Q' in $S_{2\pi-1}$ of the same kind. The points of Q' in $S_{2\pi-1}$ arise from the generators of Q on P_{12}; the points not on Q' arise from the tangents of Q on P_{12}. The group which leaves Q unaltered has the order $NC_p \div 2^{p-1}(2^p \pm 1)$, the upper or lower sign being used according as Q is an E or O quadric. Since P_{12} is any one of the set of $2^{p-1}(2^p \pm 1) - 1$ conjugate points on Q, the order of the group which leaves Q and P_{12} unaltered is $2^{2\pi} NQ'_\pi$. Clearly Q and P_{12} are unaltered by involutions I attached to all points on L_{12} which are not on Q. Moreover, in $S_{2\pi-1}$ these generate the GQ'_π. Hence the group of Q, P_{12} must contain an invariant subgroup of order $2^{2\pi}$ which consists of elements $T_{P'}$ or $S_{P'}$. If $P_{i_1 i_2 \ldots}$ is not on Q, then $P_{12 i_1 i_2 \ldots}$ is not on Q and $T_{i_1 i_2 \ldots}$ leaves Q, P_{12} unaltered. If $P_{i_1 i_2 \ldots}$ is on Q, $P_{12 i_1 i_2 \ldots}$ also is on Q, and $S_{i_1 i_2 \ldots}$ leaves Q, P_{12} unaltered. That the group of P_{12}, Q is generated by the involutions I attached to points on L_{12}, but not on Q, follows first from the fact proved above that they generate the factor group GQ'_π. Secondly they evidently generate the elements $T_{P'}$ where P' is a point of $S_{2\pi-1}$ not on Q' and, since among these points P' are 2π linearly independent points, the products of these must according to the multiplication table above, give rise to an element $T_{P'}$ or $S_{P'}$ for every point P'. But as we have seen, only elements $S_{P'}$ are admissible when P' is on Q'. Now the group of Q, P_{12} is also the group of Q, $\bar{Q}$ where $\bar{Q} = Q + P_{12}$, whence

(17) *The subgroup of GC_p which leaves two quadrics Q, $\bar{Q}$ of the same type each unaltered has the order $2^{2\pi} NQ'_\pi$, and is generated by the involutions I attached to all points not on either of the two quadrics. It has an invariant Abelian subgroup which consists of the involutions $T_{P'}$ or $S_{P'}$ according as P' is not or is a point of the projected quadric Q' in $S_{2\pi-1}$. The factor group is simply isomorphic with the group of Q'.*

We see also from what has been said above that

(18) *The group $\overline{H}_{12}$, which leaves all null lines on P_{12} unaltered, has a subgroup of order $2^{2\pi}$ corresponding to each pair Q, $\overline{Q}$ of quadrics on P_{12}. This subgroup is a regular group on the ordinary lines on P_{12}.*

Two quadrics Q, $\overline{Q}$ of different kinds, such that $Q+\overline{Q}=P_{12}$, meet in their common section by L_{12}. Each is unaltered by the involution I attached to any point not on either. But all such points are on L_{12}. Any null line on P_{12} (which is not on either quadric) touches the quadrics at a common point and contains therefore a point not on either one. Hence these involutions will generate the group GC'_{π} on the null lines of P_{12}. Moreover, I_{12} will leave every null line and also Q, $\overline{Q}$ unaltered. Hence the order of the group of Q, $\overline{Q}$ is at least $2NC'_{\pi}$. The group of Q has the order $NC_p \div 2^{p-1}(2^p \pm 1)$ and it is transitive on the quadrics $\overline{Q}$ of the other kind, whence the order of the group of Q, $\overline{Q}$ is

$$NC_p \div \{2^{p-1}(2^p \pm 1)\}\{2^{p-1}(2^p \pm 1)\} = 2NC'_{\pi}.$$

Hence

(19) *The subgroup of GC_p, which leaves two quadrics Q, $\overline{Q}$ of opposite types, such that $Q+\overline{Q}=P_{12}$ unaltered, has the order $2NC'_{\pi}$ and is generated by the involutions I attached to all points not on either quadric. It has an invariant g_2, namely $1, I_{12}$, whose factor group is simply isomorphic with GC'_{π}.*

We wish now to determine the order and nature of the subgroup $[I]$ generated by involutions $[I]$ attached to all points syzygetic with a pair of syzygetic points P_{12}, P_{34}, and therefore to P_{1234} also, excluding the generators I_{12}, I_{34}, I_{1234}. We note first from the multiplication table above that

$$I_{12} = I_{56} I_{1256} I_{78} I_{1278} I_{5678} I_{125678},$$

so that the elements excluded as generators appear in the group $[I]$. The group is an invariant subgroup of the group $H_{12,34}$, the subgroup of GC_p which leaves P_{12}, P_{34} each unaltered. The order of $H_{12,34}$ is

$$2^{2p-1}NC_{\pi} \div 2(2^{2\pi}-1) = 2^{4p-6}NC_{\pi'} \qquad (\pi' = \pi - 1),$$

since the $2(2^{2\pi}-1)$ points P_{34} syzygetic with P_{12} are conjugate under H_{12}. If we apply Theorem (16) to the points P' of $S_{2\pi-1}$ and determine thereby the group H'_{34} in $S_{2\pi-1}$ we find that the involutions $[I]$ generate a factor group of permutations of the points P', or null lines on P_{12}, whose order is $2^{2\pi-1}NC_{\pi'}$. This factor group corresponds to the invariant subgroup of the group $[I]$

made up of all elements $T_{P'}$ or $S_{P'}$ of $\overline{H}_{12}$ which leave P_{34} unaltered and which are generated by $[I]$. The pair $T_{P'}$, $S_{P'}$ leave P_{34} unaltered only if P' is one of the $(2^{2p-4}-1)-1$ points distinct from P'_{34} and syzygetic with P'_{34}. These pairs are generated by $[I]$ as are also the four additional elements $1, I_{12}$, $I_{34}I_{1234}$, $I_{12}I_{34}I_{1234}$, whence the required invariant subgroup is of order 2^{2p-8}, and the order of the group $[I]$ is $2^{4p-6}NC_{\pi'}$ and its index under $H_{12,34}$ is 2. Hence

(20) *The group generated by the involutions $[I]$ attached to all points syzygetic with two given syzygetic points, but not on the null line of the two points has the order $2^{4p-6}NC_{\pi'}$ $(\pi'=p-2)$. It has an invariant subgroup of order 2^{4p-6}, whose factor group is $GC_{\pi'}$, which effects the identical permutation on the null planes through the line on the given points. This invariant subgroup has an invariant subgroup of order 2^{2p-8} which effects the identical permutation on the null lines through one of the given points. The group $[I]$ itself is an invariant subgroup of index 2 under the group of the two points.*

Given four quadrics Q, Q', Q'', Q''' of the same type such that $Q+Q'=P_{12}$, $Q''+Q'''=P_{12}$, $Q+Q''=P_{34}$, $Q'+Q'''=P_{34}$; we ask for the order of the group generated by the involutions $[J]$ attached to points on none of the quadrics. In $S_{2\pi-1}$ the pairs Q, Q' and Q'', Q''' become quadrics R, R'' where $R+R''=P'_{34}$, and according to Theorem (17) the group generated by $[J]$ has a factor group of order $2^{2\pi'}NQ_{\pi'}$ which corresponds to an invariant subgroup of $[J]$ consisting of those elements of $\overline{H}_{12}$ which leave each of the four quadrics unaltered. These are the identity, all elements $T_{P'}$ for which P' is not on either R, R'', and all elements $S_{P'}$ for which P' is on both R' and R''. Such points P' are all points in $S_{2\pi-1}$ syzygetic with P'_{34} including P'_{34}, whence the group has the order $2^{2\pi-1}$. Hence

(21) *The group generated by the involutions $[I]$ attached to all points not on the four quadrics which arise from any one by transforming it by involutions I attached to three points on one of its generators is of order $2^{4p-7}NQ_{\pi'}$ $(\pi'=p-2)$, where $Q_{\pi'}$ is of the same type as the given quadrics. It has an invariant subgroup of order 2^{4p-7} whose factor group is isomorphic with $GQ_{\pi'}$.*

If, on the other hand, Q'', Q''' are of a type opposite to that of Q, Q' then in $S_{2\pi-1}$, R, R'' are unlike quadrics, and we apply Theorem (19) to find the group of order $2NC_{\pi'}$ in $S_{2\pi-1}$. Here the only elements of $\overline{H}_{12}$ are 1, $I_{34}I_{1234}$.

(22) *If $\overline{P_{12}, P_{34}}$ is a tangent to Q at P_{12}, the group generated by involutions I attached to all points not on the four quadrics Q, $Q+P_{12}$, $Q+P_{34}$, $Q+P_{1234}$ is of order $2^2NC_{\pi'}$ $(\pi'=p-2)$. It has an invariant subgroup made up of 1, I_{34}, I_{1234}, $I_{34}I_{1234}$ whose factor group is simply isomorphic with $GC_{\pi'}$.*

The groups described above include all which appear in the next section where the sixteen cases of $g^{(2)}_{n,k}$ are discussed.

§ 3. *Identification of the Group $g^{(2)}_{n,k}$.*

This identification will be effected by comparing the permutations of the forms b, c under the operations of $g^{(2)}_{n,k}$ which is generated by the transpositions such as T_{12} and by $A_{1,\ldots,k+1}$ with the permutations of certain sets of points or quadrics in the finite geometry of the null system under involutions I attached to certain points. In order that the isomorphism may be one-to-one it is necessary that the geometric objects in the conjugate set which correspond to $B_{1,\ldots,}B_n$ and C_0 be explicitly given; and secondly, that the conjugate set of forms b_2 be explicitly attached to a conjugate set of points. The first requirement ensures that under a given element the transforms of $x_1 \ldots, x_n$, x_0 can be uniquely determined; the second requirement ensures that the conjugate set of generating involutions can be determined since the transposition T_{12} has the invariant linear form B_{12}. So far as the other forms are concerned it is not necessary to identify them, but we shall usually do this sometimes explicitly, sometimes only so far as they occur in pairs.

In order to illustrate the easy passage from the permutation of the points to the algebraic transformation some invariant subgroups of $g^{(2)}_{n,k}$ are derived. The same division as in § 1 into sixteen cases is made, and in each of these the final description of the $g^{(2)}_{n,k}$ is given in the table (30) at the end of this section where also a reference to § 2 for a more complete description of the $g^{(2)}_{n,k}$ is made. The notation for the geometry is that of § 2 with subscripts $1, 2, \ldots, n$ with some additional subscripts selected from α, β, γ, δ.

Throughout the

CASE I: k even

we shall identify the generators as follows:

$$T_{12}=I_{12}, \quad A_{1,\ldots,k+1}=I_{1,\ldots,k+1,\alpha}.$$

For the cases $\nu=0, 1, 2, 3$ coming under $\varkappa=0$ we identify the conjugate sets of linear forms with the following sets of points in the finite geometry:

$$(23)\quad \varkappa=0 \begin{cases} \nu=0, 2: n+2 \text{ subscripts } 1, \ldots, n, \alpha, \beta. \quad b_2=P_{12},\ c_1=P_{1\alpha}; \\ b_1=P_{1\beta},\ b_0=P_{\alpha\beta};\quad b_3=P_{123\beta},\ c_2=P_{12\alpha\beta};\quad b_4=P_{1234},\ c_3=P_{123\alpha}. \\ \nu=1, 3: n+3 \text{ subscripts } 1, \ldots, n, \alpha, \beta, \gamma. \quad b_2=P_{12},\ c_1=P_{1\alpha}; \\ b_1=P_{1\beta},\ c_0=P_{\alpha\beta};\quad b_4=P_{1234},\ c_3=P_{123\alpha};\quad b_3=P_{123\beta},\ c_2=P_{12\alpha\beta}. \end{cases}$$

One difference between these cases is noteworthy. When $\nu=0, 2$ paired forms are represented by the same sets of points, but this is not serious, for we need only to identify completely the conjugate sets b_2, c_1 and b_1, c_0 in order to identify completely the conjugate set of generators, and the conjugates of the reference forms. But when $\nu=1$, the forms b_1, c_0 are paired, and when $\nu=3$ the forms b_2, c_1 are paired, and in this case the forms of a pair must be separated, as they are in fact by the above notation. It is to be understood, of course, that the notation $b_2=P_{12}$ indicates

$$B_{1,2}=P_{12},\quad B_{1,3}=P_{13},\quad B_{2,3}=P_{23}, \text{ etc.};$$
$$B_{1,2,3,4,5,6}=P_{123456}, \text{ etc.};\quad B_{1,2,\ldots,10}=P_{12\ldots10};\quad \ldots\ldots$$

We have next to verify that in each of the four cases the generators T_{12}, $A_{1,\ldots,k+1}$ permute the forms just as the corresponding involutions I_{12}, $I_{1,\ldots,k+1,\alpha}$ permute the corresponding points. The rule for permuting the forms is given in § 1 (9), that for permuting the points in § 6, 6°, 2°. The consideration of a few typical instances which will be omitted here shows that the two sets of permutations are isomorphic.

The generating involutions are the involutions I attached to all points of the sets P_{12}, $P_{1,2,\ldots,k+1,\alpha}$. The various cases now divide as follows:

$\nu=0$. The generators are attached to all points not on the quadric Q_β. Here $2p+2=n+2=4\mu+2$, $p+1=2\mu+1$ and Q_β is an E or O quadric in S_{n-1} according as $\mu\equiv0, 1$, mod. 2, or $n\equiv0, 4$, mod. 8.

$\nu=2$. The generators I are attached to all points not on the quadric Q. Here $2p+2=4\mu+4$, $p+1=2\mu+2$, and Q is an O or E quadric in S_{n-1} according as $\mu\equiv0, 1$, mod. 2, or $n\equiv2, 6$, mod. 8.

$\nu=1$. The generators are attached to all points not on the quadrics Q, $Q_{\beta\gamma}$ of opposite type. Here $2p+2=4\mu+4$, $2\pi-1=4\mu-1=n-2$.

$\nu=3$. The generators are attached to all points not on the like quadrics Q_β, Q_γ. Here $2p+2=4\mu+6$, $2\pi-1=4\mu+1=n-2$.

The salient features of the group in each of these four cases are tabulated in (30) at the end of this section, and in each case a reference to § 2 is made for a more complete description of the group.

The four cases which come under $\varkappa=2$ do not differ materially from the four just considered, and it will be sufficient to set forth for them a table analogous to (23) and to indicate in each case the position of the generators I. The details for each case can then be supplied from table (30).

$$(24)\quad \varkappa=2\left\{\begin{array}{l} \nu=0,\ 2:n+2 \text{ subscripts } 1,\ \ldots,\ n,\ \alpha,\ \beta.\quad b_2=P_{12},\ c_3=P_{123\alpha};\\ b_1=P_{1\beta},\ c_2=P_{12\alpha\beta};\quad b_3=P_{123\beta},\ c_0=P_{\alpha\beta};\quad b_4=P_{1234},\ c_1=P_{1\alpha}.\\ \nu=1,\ 3:n+3 \text{ subscripts } 1,\ \ldots,\ n,\ \alpha,\ \beta,\ \gamma.\quad b_2=P_{12},\ c_3=P_{123\alpha};\\ b_1=P_{1\beta},\ c_0=P_{\alpha\beta};\quad b_4=P_{1234},\ c_1=P_{1\alpha};\quad b_3=P_{123\beta},\ c_2=P_{12\alpha\beta}. \end{array}\right.$$

The generators I are attached to all points which,

when $\nu=0$, are not on the quadric Q_α;
when $\nu=2$, are not on the quadric $Q_{\alpha\beta}$;
when $\nu=1$, are not on the like quadrics $Q_{\alpha\beta}$, $Q_{\alpha\gamma}$;
when $\nu=3$, are not on the unlike quadrics Q_α, $Q_{\alpha\beta\gamma}$.

A reference to the table shows that when $\nu=3$, $g^{(2)}_{n,k}$ has an invariant g_2; when $\nu=1$, an invariant $g_{2^{n-1}}$. As an example of the transition from the collineations in the finite geometry to the linear transformations of the variables $x_0,\ \ldots,\ x_n$ we shall derive the equations of these invariant subgroups beginning with $\nu=3$.

According to Theorem (19) the invariant g_2 is the involution $I_{\beta\gamma}$. Under $I_{\beta\gamma}$ the point $P_{1\beta}$ becomes $P_{1\gamma}=P_{2,\ldots,n,\alpha,\beta}$, and $P_{\alpha\beta}$ becomes $P_{\alpha\gamma}=P_{1,\ldots,n,\beta}$. Hence the form B_1 becomes $c_{2,3,\ldots,n}$, and the form C_0 becomes $B_{1,\ldots,n}$. But this is precisely the effect of the linear transformation

$$x'_i=x_i+L\quad (i=0,\ 1,\ \ldots,\ n;\ L=x_0+x_1+\ldots+x_n).$$

When $\nu=1$, the null lines on $P_{\beta\gamma}$, the point common to the two quadrics, are either

$$P_{\beta\gamma},\ P_{12},\ P_{3,\ldots,n,\alpha}\quad \text{or}\quad P_{\beta\gamma},\ P_{1234},\ P_{5,\ldots,n,\alpha}.$$

On taking a section and projection from $P_{\beta\gamma}$ the subscripts β,γ are dropped and the pair of quadrics $Q_{\alpha\beta}$, $Q_{\alpha\gamma}$ becomes Q'_α in $S_{2\pi-1}$, while the null lines are either points P'_{12} or points P'_{1234} according as the number of subscripts $\not\equiv 0$ or $\equiv 0$, mod. 4. Points P'_{12} are not on Q'_α, points P'_{1234} are on Q'_α. According to Theorem (17) the elements of the invariant subgroup are either

$$I_{12}I_{3,\ldots,n,\alpha}\quad \text{or}\quad I_{1234}I_{5,\ldots,n,\alpha}I_{\beta\gamma}.$$

Taking the first as a sample of its type we find that it transforms $P_{\alpha\beta}$, $P_{1\beta}$, $P_{3\beta}$ into $P_{3,\ldots,n,\beta}$, $P_{2\beta}$, $P_{4,\ldots,n,\alpha,\beta}$, and therefore sends C_0, B_1, B_3 into $B_{3,\ldots,n}$, B_2, $C_{4,\ldots,n}$ respectively, whence it is

$$x_0'=x_0+C_{3,\ldots,n},\quad x_i'=x_i+B_{1,2},\quad x_j'=x_j+C_{3,\ldots,n}\qquad (i=1,2;\ j=3,\ldots,n).$$

The general element of this type is obtained from this by shifting sets of four subscripts from the form C to the form B and is

$$x_0'=x_0+C,\quad x_i'=x_i+B,\quad x_j'=x_j+C$$
$$(i=1,\ldots,4c+2;\ j=4c+3,\ldots,n)$$
$$B=B_{1,2,\ldots,4c+2},\quad C=C_{4c+3,\ldots,n}.$$

Similarly we find that the general element of the other type is

$$x_0'=x_0+B,\quad x_i'=x_i+C,\quad x_j'=x_j+B$$
$$(i=1,\ldots,4c;\ j=4c+1,\ldots,n)$$
$$B=B_{1,\ldots,4c},\quad C=C_{4c+1,\ldots,n}.$$

We take up now the

CASE II: k odd.

There is a sharp difference between the cases $\varkappa=1$ and $\varkappa=3$. When k is odd the generators § 1 (4) generate a group $g'^{(2)}_{n,k}$ on the variables $x_1,\ldots,x_n$ alone, and $g'^{(2)}_{n,k}$ is necessarily isomorphic with $g^{(2)}_{n,k}$. In the case $\varkappa=3$ there is according to § 1 (7), an invariant quadratic form which contains the variable x_0, and from the invariance of this form the effect of any element of $g^{(2)}_{n,k}$ upon x_0 can be found when its effect upon the variables $x_1,\ldots,x_n$ is known. Hence in this case the group $g'^{(2)}_{n,k}$ is simply isomorphic with $g^{(2)}_{n,k}$. However, in the case $\varkappa=1$ there may be an invariant subgroup of order μ of $g^{(2)}_{n,k}$ whose elements have the form

$$x_0'=x_0+F_j,\quad x_i'=x_i\qquad (i=1,\ldots,n;\ j=1,\ldots,\mu),$$

and then $g^{(2)}_{n,k}$ is in $\mu-1$ isomorphism with $g'^{(2)}_{n,k}$.

We shall therefore consider these cases separately and begin with

CASE II 1: $\varkappa=1$,

and determine first the group $g'^{(2)}_{n,k}$. According to table (14) we have now four conjugate sets of forms b_1, b_2, b_3, b_4, and the sets b_1, b_2 must be completely identified in order to determine respectively the linear transformations and the conjugate generators. We shall take for

$\nu=0, 2$: subscripts $1, 2, \ldots, n, \alpha, \beta$,
$\nu=1, 3$: subscripts $1, 2, \ldots, n, \alpha$.

We identify the generators and forms b as follows:

(25) $T_{12}=I_{12}$, $A_{1,\ldots,k+1}=I_{1,\ldots,k+1}$; $b_2=P_{12}$; $b_1=P_{1\alpha}$; $b_3=P_{123\alpha}$; $b_4=P_{1234}$.

Then from § 1 (12) we find that these linear forms are permuted by the generators precisely as the points are permuted by the involutions I, and we have only to identify the location of the generators.

$\nu=0$. The generators belong to all points not on the like quadrics Q_α, Q_β. Here $2p+2=4\mu+2$, $p+1=2\mu+1$, $2\pi-1=n-3$; and Q_α, Q_β are E or O quadrics according as $\mu\equiv 0, 1$, mod. 2.

$\nu=2$. The generators belong to all points not on the unlike quadrics Q, $Q_{\alpha\beta}$. Here $2p+2=4\mu+4$, $2\pi-1=n-3$.

$\nu=1$. The generators belong to all points not on Q_α. Here $2p+2=4\mu+2$, $p+1=2\mu+1$, and Q_α is an E or O quadric according as $\mu\equiv 0, 1$, mod. 2.

$\nu=3$. The generators belong to all points not on Q. Here $2p+2=4\mu+4$, $p+1=2\mu+2$, and Q is an E or O quadric according as $\mu\equiv 1, 0$, mod. 2.

These facts concerning $g'^{(2)}_{n,k}$ are collected in the table:

(26) $g'^{(2)}_{n,k} : \varkappa=1$

ν	Order of Invariant Subgroup	Factor Group	Invariant Quadric $n\equiv\nu'$, mod. 8	Reference
0	2^{n-2}	GQ_{n-3}	$Q=\frac{E}{O}$ if $\nu'=\frac{0}{4}$	§ 2 (17)
2	2	GC_{n-3}		§ 2 (19)
1		GQ_{n-2}	$Q=\frac{E}{O}$ if $\nu'=\frac{1}{5}$	§ 2, 7°
3		GQ_{n-2}	$Q=\frac{E}{O}$ if $\nu'=\frac{7}{3}$	§ 2, 7°

In order to obtain a representation which will separate the forms C, we will use when

$$\nu=0,\,2 : \text{subscripts } 1, 2, \ldots, n, \alpha, \beta, \gamma, \delta;$$
$$\nu=1,\,3 : \text{subscripts } 1, 2, \ldots, n, \alpha, \beta, \gamma;$$

and in all four of these cases will identify the generators and the forms C as follows:

(27) $T_{12}=I_{12}$, $A_{1,2,\ldots,k+1}=I_{1,2,\ldots,k+1,\alpha,\beta}$; $c_1=P_{1\alpha}$; $c_3=P_{123\beta}$; $c_0=P_{\alpha\gamma}$; $c_2=P_{12\beta\gamma}$.

Then the generators affect the forms c as is required in § 1 (9). Moreover, the behavior of the forms b can be deduced from that of the forms c for $B_1=C_0+C_1$. Taking up the four cases in order we find that when:

$\nu=0$. The generators belong to all points not on any one of the four like quadrics $Q_{\alpha\gamma}$, $Q_{\alpha\delta}$; $Q_{\beta\gamma}$, $Q_{\beta\delta}$, and the group is described in § 2 (21). Here $2p+2=4\mu+4=n+4$, $p+1=2\mu+2$ and the quadrics are of type E or O, according as $\mu\equiv 0, 1$, mod. 2.

$\nu=2$. The generators belong to all points not on any of the pairs of unlike quadrics Q_α, Q_β, $Q_{\alpha\gamma\delta}$, $Q_{\beta\gamma\delta}$. Here $2p+2=4\mu+6=n+4$.

$\nu=1$. The generators belong to all points not on the pair of like quadrics $Q_{\alpha\gamma}$, $Q_{\beta\gamma}$. Here $2p+2=4\mu+4=n+3$, $p+1=2\mu+2$ and the quadrics are E or O quadrics according as $\mu\equiv 0, 1$, mod. 2.

$\nu=3$. The generators belong to all points not on the like quadrics Q_α, Q_β. Here $2p+2=4\mu+6=n+3$, $p+1=2\mu+3$ and the quadrics are of type O, E, according as $\mu\equiv 0, 1$.

Thus a comparison of these results as listed in table (30) with the table (26) above, shows that $g^{(2)}_{n,k}$ has an invariant G_μ whose factor group is $g'^{(2)}_{n,k}$, and that $\mu=2^{n-1}$ except in the case $\nu=2$ for which $\mu=2$. When $\mu=2^{n-1}$ the subgroup G_μ must consist of elements of the form

$$x_0'=x_0+F,\ x_i'=x_i\quad (i=1,\ldots,n)\quad \text{where } F=x_{i_1}+x_{i_2}+\ldots+x_{i_{2j}}\quad (0\overline{\gtrless}2j\overline{\gtrless}n).$$

When $\mu=2$ G_μ consists of the identity and the single element $F=L$ the invariant linear form.

We take up finally the

CASE II 2 : $x=3$.

If in this case the forms c are completely identified, the transformations are determined. For all values of ν we take

$$(28)\qquad T_{12}=I_{12},\ A_{1,\ldots,k+1}=I_{1,\ldots,k+1};\ b_2=P_{12},\ b_4=P_{1234}.$$

Then for various values of ν we represent the forms c as follows:

(29)

ν	Subscripts $1,\ldots,n$ and	c_0	c_1	c_2	c_3
0	α,β	Q_α	Q_1	$Q_{12\alpha}$	Q_{123}
2	α,β	$Q_{\alpha\beta}$	$Q_{1\alpha}$	$Q_{12\alpha\beta}$	$Q_{123\alpha}$
1	α,β,γ	$Q_{\beta\gamma}$	$Q_{1\beta}$	$Q_{12\beta\gamma}$	$Q_{123\beta}$
3	α,β,γ	Q_α	$Q_{1\alpha\gamma}$	$Q_{12\alpha}$	$Q_{123\alpha\gamma}$

$\nu=0, 2$. The generators belong to all points syzygetic with $P_{\alpha\beta}$. Here $2p+2=4\mu+2=n+2$ and $S_{2\pi-1}=S_{n-3}$.

$\nu=1, 3$. The generators belong to all points syzygetic with the azygetic triad $P_{\beta\gamma}$, $P_{\gamma\alpha}$, $P_{\alpha\beta}$. Here $2p+2=4\mu+4=n+3$ and $S_{2\pi-1}=S_{n-2}$.

This completes the discussion of the sixteen cases, and the structure and orders of the various types of $g^{(2)}_{n,k}$ are readily ascertained from the table (30).

(30)

	ν	Order of Invariant Subgroup	Factor Group*	Invariant Quadric $n\equiv\nu'$, mod. 8	Reference to § 2
$\varkappa=0$	0		GQ_{n-1}	$Q={E \atop O}$ if $\nu'={0 \atop 4}$	7°
	1	2	GC_{n-2}		(19)
	2		GQ_{n-1}	$Q={E \atop O}$ if $\nu'={6 \atop 2}$	7°
	3	2^{n-1}	GQ_{n-2}	$Q={E \atop O}$ if $\nu'={7 \atop 3}$	(17)
$\varkappa=1$	0	$2^{n-2}.\,2^{n-1}$	GQ_{n-3}	$Q={E \atop O}$ if $\nu'={0 \atop 4}$	(21)
	1	2^{n-1}	GQ_{n-2}	$Q={E \atop O}$ if $\nu'={1 \atop 5}$	(17)
	2	$2\,.\,2$	GC_{n-3}		(22)
	3	2^{n-1}	GQ_{n-2}	$Q={E \atop O}$ if $\nu'={7 \atop 3}$	(17)
$\varkappa=2$	0		GQ_{n-1}	$Q={E \atop O}$ if $\nu'={0 \atop 4}$	7°
	1	2^{n-1}	GQ_{n-2}	$Q={E \atop O}$ if $\nu'={1 \atop 5}$	(17)
	2		GQ_{n-1}	$Q={E \atop O}$ if $\nu'={2 \atop 6}$	7°
	3	2	GC_{n-2}		(19)
$\varkappa=3$	0	2^{n-1}	GC_{n-3}		(16)
	1		GC_{n-2}		6°, (15)
	2	2^{n-1}	GC_{n-3}		(16)
	3		GC_{n-2}		6°, (15)

[In this table the subscript of Q and C is the dimension of the space in which the quadric or null system lies.]

It is proved in P. S. II (27) that the groups $g_{n,k}$ and $g_{n,n-k-2}$ are simply isomorphic and a transformation T which sends the one group into the other is given there. This transformation is degenerate modulo 2, so that it is not necessarily true that the modular groups $g^{(2)}_{n,k}$ and $g^{(2)}_{n,n-k-2}$ are simply isomorphic. We should expect, however, that some sort of isomorphism persists, and this is found to be the case. In the table below the distinct associated cases are opposite each other. We find from (30) that either the

* These factor groups are discussed in full in Dickson's "Linear Groups," Chapters II, VIII.

associated groups are simply isomorphic, or one has an invariant subgroup (noted on the margin) whose factor group is simply isomorphic with the other.

(31)

		$\varkappa,\ \nu$	$\varkappa'=\nu-\varkappa-2,\ \nu$	
		0, 0	2, 0	
	g_2	0, 1	3, 1	
		0, 3	1, 3	
		1, 1	2, 1	
		1, 2	3, 2	$g_{2^{n-3}}$
	g_2	2, 3	3, 3	

We are above all interested in the sets P^p_{2p+2}, which may be required to be self-associated. The four cases in question here abstracted from (30) are:

(32)

p mod. 4	$2p+2$ mod. 8	Invariant Subgroup	Factor Group
0	2		GO_{2p+1}
1	4	$G_2 4p+1$	GO_{2p-1}
2	6		GO_{2p+1}
3	0	$G_2 2p+1$	GC_{2p-1}

We shall show in the next section that the requirement that the set P^p_{2p+2} be self-associated does not reduce the group $g^{(2)}_{2p+2,\,p}$ when p is odd; but that when p is even this requirement reduces the group to one with an invariant G_2 whose factor group is GC_{2p-1}. Thus in all cases the self-associated set defines a modular group in the finite geometry of S_{2p-1}. In order to impose the conditions for self-association most conveniently, it is necessary to consider the ultra-elliptic set P^p_{2p+2}.

§ 4. *Modular Groups Determined by $e_{n,k}$.*

Congruent ultra-elliptic sets P^k_n, P'^k_n lie on projectively equivalent norm-elliptic curves E^{k+1}, E'^{k+1}. When E'^{k+1} is projected upon E^{k+1}, P'^k_n is projected upon a set on E^{k+1} whose elliptic parameters $u'_1, \ldots, u'_n$ are expressed in terms of the elliptic parameters $u_1, \ldots, u_n$ of P^k_n by means of a linear transformation—an element of the group $e_{n,k}$. The generators of $e_{n,k}$ are the transpositions of the u's and the element $A_{1,\ldots,k+1}$ which is (P. S. II (32))

$$A_{1,\ldots,k+1}:\quad \begin{aligned} u'_i&=u_i-\frac{2}{k+1}(u_1+\ldots+u_{k+1}) \qquad (i=1,\ldots,k+1),\\ u'_j&=u_j+\frac{k-1}{k+1}(u_1+\ldots+u_{k+1}) \qquad (j=k+2,\ldots,n). \end{aligned}$$

All the elements of $e_{n,k}$ have for coefficients rational numbers with denominators which are factors of $k+1$ or of $\dfrac{k+1}{2}$ according as k is even or odd [P. S. II (37)].

Let us consider the effect of these elements upon linear forms,

$$x_1u_1+x_2u_2+\dots+x_nu_n,$$

with integer coefficients $x_1, \dots, x_n$. This is transformed by $A_{1,\dots,k+1}$ into the form

$$x'_1u'_1+x'_2u'_2+\dots+x'_nu'_n,$$

where

$$x'_i=x_i+\frac{k-1}{k+1}\sum_1^n x_h-(x_1+\dots+x_{k+1}) \qquad (i=1, \dots, k+1),$$
$$x'_j=x_j \qquad (j=k+2, \dots, n).$$

Hence

$$\sum_1^n x'_h=\sum_1^n x_h+(k-1)\sum_1^n x_h-(k+1)(x_1+\dots+x_{k+1}).$$

If therefore

$$(33) \qquad \sum_{h=1}^n x_h=(k+1)x_0 \ \ (k \text{ even}), \ \text{ or } \ \sum_{h=1}^n x_h=\frac{k+1}{2}x_0 \ \ (k \text{ odd}),$$

where x_0 is an integer, then also

$$\sum_1^n x'_h=(k+1)x'_0 \ \ (k \text{ even}), \ \text{ or } \ \sum_1^n x'_h=\frac{k+1}{2}x'_0 \ \ (k \text{ odd}),$$

where x'_0 is also an integer. If we set

$$(34) \qquad y_0=x_0 \ \ (k \text{ even}), \ \text{ or } \ y_0=\frac{x_0}{2} \ \ (k \text{ odd}),$$

we find in either case that

$$(35) \quad \begin{cases} y'_0=y_0+\{(k-1)y_0-(x_1+\dots+x_{k+1})\}, \\ x'_i=x_i+\{(k-1)y_0-(x_1+\dots+x_{k+1})\} \qquad (i=1, \dots, k+1), \\ x'_j=x_j \qquad (j=k+2, \dots, n). \end{cases}$$

This linear transformation (35) is precisely the generator $A_{1,\dots,k+1}$ of the group $g_{n,k}$. Hence

(36) *Under the group $e_{n,k}$ linear forms with integer coefficients x which satisfy the relations* (33) *are permuted contragrediently to integer linear forms under the group $g_{n,k}$.*

Hence within the totality of integer linear forms defined by (33) we shall have a modular theory identical with that of $g_{n,k}$ which has been discussed in the preceding sections for the modulus 2.

In the case of the ultra-elliptic set P^p_{2p+2} the single condition for self-association is

$$u_1+u_2+\dots+u_{2p+2}=0.$$

This is a linear form included in the aggregate (33). It corresponds to the point 0, 1, 1, ..., 1 for the group $g^{(2)}_{n,k}$ or through the medium of the invariant polarized form M to the linear form $B_{1,2,\ldots,n}$. When p is odd, $B_{1,\ldots,n}$ is the invariant form L so that the requirement $B_{1,\ldots,n}=0$ leads to no change in the group $g^{(2)}_{2p+2,p}$. If, however, p is even, the form $B_{1,\ldots,n}$ (see cases $x=0$, $\nu=2$; $x=2$, $\nu=2$) is represented in the finite geometry by the point $P_{\alpha\beta}$, and we ask for the subgroup of $g^{(2)}_{2p+2,p}$ which leaves $P_{\alpha\beta}$ unaltered. When $p\equiv 0$, mod. 4, there is an invariant O quadric Q not on $P_{\alpha\beta}$ so that the group reduces to that of the O, E quadrics Q, $Q_{\alpha\beta}$. When $p\equiv 2$, mod. 4, there is an invariant O quadric $Q_{\alpha\beta}$ not on $P_{\alpha\beta}$ so that the group reduces to that of the O, E quadrics $Q_{\alpha\beta}$, Q. In either case the group is reduced from a GO_{2p+1} to a group with an invariant G_2 whose factor group is a GC_{2p-1}. Now if a group H has an invariant subgroup I with factor group F, and if F has an invariant subgroup I' with factor group F', then H has a larger invariant subgroup I'' which contains I whose factor group is F'. Hence we can state that

(37) *The group $g_{2p+2,p}$ (p odd) has an invariant subgroup whose factor group is GC_{2p-1} if $p\equiv 3$, mod. 4; or GO_{2p-1} if $p\equiv 1$, mod. 4. If p is even the group $g_{2p+2,p}$ has a subgroup g' which consists of those elements which leave $x_1+\ldots+x_{2p+2}$ unaltered, mod. 2, and g' has an invariant subgroup whose factor group is GC_{2p-1}.*

The above theorem concerning $g_{2p+2,p}$ can be translated at once to apply to the simply isomorphic Cremona group $G_{2p+2,p}$ which is projectively attached to the point set P^p_{2p+2}, the subgroup g' of the theorem being that which is determined by the projective conditions for self-association. Thus the fact that the self-associated set defines groups which are isomorphic with those of the half periods of the theta functions in p variables confirms the existence of a connection between the absolute invariants of the self-associated set and the theta modular functions. In case $p\equiv 1$, mod. 4, there is indicated further that in this connection an odd theta-characteristic is isolated.

BALTIMORE, *April* 15, 1918.

On the Asymptotic Solution of the Non-Homogeneous Linear Differential Equation of the n-th Order. A Particular Solution.

By W. Van N. Garretson.

The asymptotic development* for the irregular integrals of a homogeneous linear differential equation has been obtained by both Horn† and Love.‡ Horn has published several papers on the case where the roots of the characteristic equation are all distinct, while Love has taken up the case where the roots of the characteristic equation are unrestricted as to their order of multiplicity, including the case of distinct roots as a special case.

In this paper we shall consider the non-homogeneous equation where the roots of the characteristic equation are distinct, and follow, at the outset, the method employed by Dini § in his researches on linear differential equations. In Section I the two theorems stated and proved by him will be generalized so as to apply to the non-homogeneous equation and combined in one theorem. In Section II we shall determine a particular solution of the given equation. To this end we shall make use of the researches of Love ‡ in the homogeneous linear differential equation by employing his solutions. The particular solution thus obtained of the non-homogeneous equation will be in the form of quadratures. The determination of the asymptotic development of the particular integral found in Section II will form the content of Section III.

Section I.

Take for consideration the non-homogeneous linear differential equation

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\dots a_n(x)y=X(x), \tag{1}$$

* "Asymptotic Development in Poincaré's Sense," cf. *Acta Mathematica*, Vol. VIII (1886), p. 297.

† *Journal für Mathematik*, Vol. CXXXVIII (1910), pp. 159–191.

‡ *Annals of Mathematics*, Second Series, Vol. XV (1914), pp. 145–156. Also American Journal of Mathematics, Vol. XXXVI, No. 2 (1914), pp. 151–166.

§ *Annali di Matematica*, Ser. 3, Vol. II (1898), pp. 297–324. *Ibid.*, Vol. III (1899), pp. 125–183. Important contributions have also been made by Poincairé, Kneser, Birkhoff, and others.

in which the coefficients are real or complex functions developable, asymptotically, for large values of x in the form

$$a_i(x) \sim x^{ik}\left[a_{i,0}+\frac{a_{i,1}}{x}+\frac{a_{i,2}}{x^2}+\ldots\right]; \quad i=1, 2, \ldots, n; \quad k=0, 1, 2, \ldots,$$

while the first $n-i$ derivatives also possess asymptotic developments. The function $X(x)$ will be considered as capable of asymptotic development in the form

$$X(x) \sim x^m\left[b_0+\frac{b_1}{x}+\frac{b_2}{x^2}+\ldots\right],$$

where $m, b_0, b_1, b_2, \ldots,$ are real or complex quantities and $b_0 \neq 0$.

We regard y for the present as a known solution of (1). Let us choose n auxiliary functions $z_1, \ldots, z_n$ of x, which, with their first n derivatives, are continuous for large values of x, and such that for the same values of x the determinant

$$Q(x)=\begin{vmatrix} z_1 & z_1' & z_1'' & z_1''' \ldots z_1^{(n-1)} \\ z_2 & z_2' & z_2'' & \ldots\ldots z_2^{(n-1)} \\ \ldots & \ldots & \ldots & \ldots \\ z_n & z_n' & z_n'' & \ldots\ldots z_n^{(n-1)} \end{vmatrix} \tag{2}$$

never vanishes. Place

$$\int_{\beta_i}^{x}[y^{(n)}+a_1(x_1)y^{(n-1)}+a_2(x_1)y^{(n-2)}+\ldots+a_n(x_1)y]z_i(x_1)\,dx_1$$
$$=\int_{\beta_i}^{x}X(x_1)z_i(x_1)\,dx_1; \quad i=1, \ldots, n \tag{3}$$

where β_i are sufficiently large, positive, real quantities, or infinity, and x takes on large real values.* It is understood that, for the interval of integration, all the functions here appearing including y and its first n derivatives are continuous.

An integration by parts gives

$$p_{i,0}y^{(n-1)}+p_{i,1}y^{(n-2)}+p_{i,2}y^{(n-3)}+\ldots+p_{i,n-1}y$$
$$=\int_{\beta_i}^{x}[y(x_1)Z_i(x_1)+X(x_1)z_i(x_1)]\,dx_1+c_i, \tag{4}$$

where

$$\left.\begin{aligned} &p_{i,0}=z_i, \quad p_{i,1}=z_ia_1-z_i'=z_ia_1-p_{i,0}', \ldots, \quad p_{i,n-1}=z_ia_{n-1}-p_{i,n-2}', \\ &-Z_i=z_ia_n-p_{i,n-1}'=z_ia_n+\varepsilon_1(z_ia_{n-1})'+\varepsilon_2(z_ia_{n-2})''+\ldots+\varepsilon_nz_i^{(n)}, \\ &c_i=[p_{i,0}y^{(n-1)}+p_{i,1}y^{(n-2)}+\ldots+p_{i,n-1}y]_{x=\beta_i}; \\ &\varepsilon_i=(-1)^i; \quad i=1, \ldots, n. \end{aligned}\right\} \tag{5}$$

*In treating the homogeneous equation, Dini takes $\beta_1=\beta_2=\ldots=\beta_n$, while Horn, in a similar discussion, places, as convenience demands, $\beta_1=\beta_2=\ldots=\beta_v=\infty$ and $\beta_{v+1}=\ldots=\beta_n=\beta$ where $0 \leq v \leq n$ and β is real and positive. The second method is the only one which seems applicable, to the problem in hand, in order to insure the convergence of all of the integrals.

For the values of x under consideration the determinant

$$P(x) = \begin{vmatrix} p_{1,0} & p_{1,1} \cdots \cdot p_{1,n-1} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ p_{n,0} & p_{n,1} \cdots \cdot p_{n,n-1} \end{vmatrix}$$

never vanishes, because

$$P(x) = (-1)^{\frac{n(n-1)}{2}} Q(x).$$

The value of y from equations (4) is found to be

$$y(x) = g(x) + \sum_{i=1}^{n} \int_{\beta_i}^{x} (y(x_1) k_i(x, x_1) + X(x_1) K_i(x, x_1)) \, dx_1, \tag{6}$$

where

$$g(x) = \frac{A(x)}{Q(x)}; \quad A(x) = \begin{vmatrix} z_1 & z_1' \ldots z_1^{(n-2)} & c_1 \\ z_2 & z_2' \ldots z_2^{(n-2)} & c_2 \\ \cdots & \cdots\cdots\cdots & \cdots \\ z_n & z_n' \ldots z_n^{(n-2)} & c_n \end{vmatrix},$$

$$k_i(x, x_1) = \frac{Z_i(x_1) q_i(x)}{Q(x)} \quad \text{and} \quad K_i(x, x_1) = \frac{z_i(x_1) q_i(x)}{Q(x)},$$

$q_i(x)$ is the cofactor with respect to the *i-th* row of the last column of $Q(x)$.

The formula (6) may be used as a recursion formula. By repeated application of the recursion formula the value of y as given in (6) can be expressed in the form of an infinite series as follows:

$$y(x) = \sum_{\lambda=0}^{\infty} \mu_\lambda(x) + \tau_\lambda(x), \tag{7}$$

where

$$\mu_\lambda(x) = \sum_{i=1}^{n} \int_{\beta_i}^{x} k_i(x, x_1) \, dx_1 \cdot \sum_{i=1}^{n} \int_{\beta_i}^{x_1} k_i(x_1, x_2) \, dx_2 \ldots \cdot \sum_{i=1}^{n} \int_{\beta_i}^{x_{\lambda-1}} k_i(x_{\lambda-1}, x_\lambda) g(x_\lambda) \, dx_\lambda,$$

$$\tau_\lambda = \sum_{i=1}^{n} \int_{\beta_i}^{x} k_i(x, x_1) \, dx_1 \cdot \sum_{i=1}^{n} \int_{\beta_i}^{x_1} k_i(x_1, x_2) \, dx_2 \ldots \sum_{i=1}^{n} \int_{\beta_i}^{x_{\lambda-2}} k_i(x_{\lambda-2}, x_{\lambda-1}) \, dx_{\lambda-1}$$

$$\cdot \sum_{i=1}^{n} \int_{\beta_i}^{x_{\lambda-1}} K_i(x_{\lambda-1} x_\lambda) X(x_\lambda) \, dx_\lambda, \qquad \lambda = 1, 2, 3, \ldots .$$

In the above $\mu_0(x) = g(x)$ and $\tau_0(x) = 0$.

The expression for y as given in (7) is still regarded as a known solution of (1). The form of the solution being here obtained, we can proceed at once to state how an unknown solution could be built up. Choose n auxiliary functions $z_1, z_2, \ldots, z_n$ of x which, with their first n derivatives, are continuous for large, positive, real values of x, and such that for the same values of x the determinant $Q(x)$ as given in (2) never vanishes. The functions $Z_1, \ldots, Z_n$ of x can now be formed according to (5); also $A(x)$, $g(x)$, $k_i(x, x_1)$ and

$K_i(x, x_1)$ as given in (6). In building up $A(x)$ it should be noted that $c_1, \ldots, c_n$ are now arbitrary constants and not functions of the betas. To simplify the work choose all the c's equal to zero except c_r. The function $g(x)$ will become $g_r(x) = \dfrac{c_r A_r(x)}{Q(x)}$ where $A_r(x)$ is the cofactor with respect to c_r in $A(x)$. Represent the corresponding values of $\mu_\lambda(x)$ and $y(x)$ by $\mu_{\lambda,r}(x)$ and $y_r(x)$ respectively.

We shall now state and prove the following:

THEOREM I. *Suppose that a large, positive, real number β can be found such that for the values of β_i either equal to β or ∞, and for all values of x greater than β the series*

$$y_r(x) = \sum_{\lambda=0}^{\infty} (\mu_{\lambda,r}(x) + \tau_\lambda(x)) \tag{8}$$

satisfies the following conditions:

(a) The series $\Sigma\mu_{\lambda,r}(x)$ and $\Sigma\tau_\lambda(x)$ converge.

(b) The series for $y_r(x)$ when multiplied by $k_i(x, x_1)$ may be integrated term by term with respect to x_1 from β to x, or from x to ∞ in case β_i equals ∞ $i=1, \ldots, v$ where $0 \leqq v \leqq n$.

(c) The series (8) defines a function $y_r(x)$ such that each of the integrals $\int_\beta^x y_r(x) Z_i(x)\,dx$ and $\int_\beta^x z_i(x) X(x)\,dx$ (or $\int_x^\infty y_r(x) Z_i(x)\,dx$ and $\int_x^\infty z_i(x) X(x)\,dx$ in case $\beta_i = \infty$; $i=1, \ldots, v$ where $0 \leqq v \leqq n$) has a meaning when $x > \beta$.

Then for such values of x the function $y_r(x)$ is an integral of (1).

PROOF: The values of $p_{i,k}$, $i=1, \ldots, n$; $k=1, \ldots, n-1$ are given in (5). Place

$$\phi_s(x) = \int_{\beta_s}^x y_r(x) Z_s(x)\,dx, \quad s=1, 2, \ldots, r-1, r+1, \ldots, n,$$

$$\phi_r(x) = \int_{\beta_r}^x y_r(x) Z_r(x)\,dx + c_r; \quad \psi_s(x) = \int_{\beta_s}^x z_s(x) X(x)\,dx, \quad s=1, \ldots, n,$$

$$\Delta_r(x) = \begin{vmatrix} p_{1,0} & p_{1,1} \cdots p_{1,n-2} & \phi_1 + \psi_1 \\ \cdots & \cdots & \cdots \\ p_{n,0} & p_{n,1} \cdots p_{n,n-2} & \phi_n + \psi_n \end{vmatrix};$$

$$\Delta(x) = \begin{vmatrix} p_{1,0} & p_{1,1} \cdots\cdots & p_{1,n-1} \\ \cdots & \cdots & \cdots \\ p_{n,0} & p_{n,1} \cdots\cdots & p_{n,n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} Q(x).$$

Now by condition (c) the series (8) may be written

$$y_r(x)=g_r(x)+\sum_{i=1}^{n}\int_{\beta_i}^{x}(k_i(x,x_1)y_r(x_1)+K_i(x,x_1)X(x_1))\,dx_1. \tag{9}$$

By substituting the values of $g_r(x)k_i(x,x_1)$ and $K_i(x,x_1)$ in (9), and taking account of equations (3) we find that

$$y_r(x)=\Delta_r(x)/\Delta(x),$$

so it suffices for our proof to show that this function is an integral (1).

To do this consider the system of n functions $\eta_0, \eta_1, \ldots, \eta_{n-1}$, each defined for all values of x sufficiently large by means of the following system of n linear equations:

$$p_{s,0}\eta_{n-1}+p_{s,1}\eta_{n-2}+p_{s,2}\eta_{n-3}+\ldots+p_{s,n-1}\eta_0=\phi_s+\psi_s, \qquad s=1,2,\ldots,n. \tag{10}$$

It follows that

$$\eta_0=\Delta_r(x)/\Delta(x)=y_r(x). \tag{11}$$

By differentiating (10) with respect to x and making use of (5) and (11), we find that

$$z_s(\theta-X)+p'_{s,0}\theta_1+p'_{s,1}\theta_2+p'_{s,2}\theta_3+\ldots+p'_{s,n-2}\theta_{n-1}=0, \qquad s=1,\ldots,n, \tag{12}$$

where

$$\theta=\eta'_{n-1}+a_1(x)\eta'_{n-2}+a_3(x)\eta'_{n-3}+\ldots+a_{n-1}(x)\eta'_0+a_n(x)\eta_0, \tag{13}$$

$$\theta_1=\eta_{n-1}-\eta'_{n-2};\quad \theta_2=\eta_{n-2}-\eta'_{n-3};\quad \ldots;\quad \theta_{n-1}=\eta_1-\eta'_0. \tag{14}$$

The system (13) consists of n homogeneous equations in n unknowns. Upon noting that $p'_{d,b}=z_d a_{b+1}-p_{d,b+1}\begin{cases}d=1,\ldots,n\\ b=0,1,\ldots,n-2\end{cases}$ the discriminant of the system reduces at once to $(-1)^{n-1}Q(x)$, and hence does not vanish for any values of x under consideration, whence

$$\theta-X=\theta_1=\theta_2=\ldots=\theta_{n-1}=0,$$

or, by (11) and (14),

$$\eta_s=y_r^{(s)}, \qquad s=1,2,\ldots,n.$$

Substituting these values in (13), we find

$$y_r^{(n)}+a_1(x)y_r^{(n-1)}+a_2(x)y_r^{(n-2)}+\ldots+a_n(x)y_r=X(x),$$

which was to be proved.

Section II.

In this section we shall endeavor to choose the auxiliary functions $z_1\ldots z_n$ in such a way that the above theorem may be employed to obtain a particular solution of (1). The complementary function corresponding to (1), *i. e.*, the general solution of

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\ldots+a_n(x)y=0 \tag{15}$$

has already been obtained by Horn and Love. We shall suppose the characteristic equation of (1), viz.,

$$w^n + a_{1,0}w^{n-1} + a_{2,0}w^{n-2} + \ldots + a_{n,0} = 0 \tag{16}$$

to have n distinct roots $w_1, \ldots, w_n$. Consider, now, the functions $Z_i(x_1)$ which occur in $k_i(x, x_1)$, $i=1, \ldots, n$. These functions are all of the form of the left member of the differential equation

$$a_n z - a_{n-1}z' + a_{n-2}z'' - a_{n-3}z''' + \ldots + (-1)^n z^{(n)} = 0, \tag{18}$$

an equation of the same type as (15). A fundamental system of solutions of (18) will be, according to the results obtained by Love,*

$$z_i(x) = e^{-f_i(x)} x^{-a_{i,0}} P_{i,0}(x), \qquad i=1, \ldots, n, \tag{19}$$

where

$$f_i(x) = w_i \frac{x^{k+1}}{k+1} + \alpha_{i,-k}\frac{x^k}{k} + \alpha_{i,-k+i}\frac{x^{k-1}}{k-1} + \ldots + \alpha_{i,-1}x, \tag{20}$$

and $P_{i,0}$ is of the form†

$$P_{i,0}(x) = \delta_{i,0} + \frac{\delta_{i,1}}{x} + \frac{\delta_{i,2}}{x^2} + \ldots + \frac{\delta_{i,p} + \varepsilon_{i,p}(x)}{x^p}, \qquad i=1, \ldots, n,$$

where p is an arbitrary positive integer and $\lim\limits_{x=\infty} \varepsilon_{i,p}(x) = 0$.

By choosing the auxiliary functions as in (19), viz., the solutions of (18), the functions $Z_i(x)$ vanish, and this assures the vanishing of $k_i(x, x_1)$. The value of $y_r(x)$ as found in (8) then becomes

$$y_r(x) = g_r(x) + \sum_{i=1}^{n} \int_{\beta_i}^{x} K_i(x, x_1) X(x_1)\, dx_1. \tag{21}$$

The first term of the right member of (21), viz., $g_r(x)$, is a particular solution of the homogeneous differential equation (15). To obtain a particular solution of (1) set c_r in g_r equal to zero, thus making g_r vanish. Then we have as a particular solution of (1),

$$y(x) = \sum_{i=1}^{n} \int_{\beta_i}^{x} K_i(x, x_1) X(x_1)\, dx_1, \qquad i=1, \ldots, n, \tag{22}$$

provided the conditions of Theorem I are satisfied. Of these three conditions, (a) is satisfied inasmuch as the series reduces to a single term; (b) is satisfied because the functions $k_i(x, x_1)$, $i=1, \ldots, n$ vanish. It will be evident that (c) is satisfied when we obtain the asymptotic development for the right member of (22).

---

* *Annals of Mathematics*, Second Series, Vol. XV (1914), pp. 145–156. Also AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVI, No. 2 (1914), pp. 151–166.

† In this paper the functions $P(x)$, generally written with subscripts, will have the form of $P_{i,0}(x)$ here given.

By taking the auxiliary functions as above stated, and replacing $X(x)$ by its value, the expression for $y(x)$ in (22) becomes

$$y(x)=\sum_{i=1}^{n}\int_{\beta_i}^{x} e^{f_i(x)}x^{a_{i,0}-(n-1)k}P_{i,1}(x)\,e^{-f_i(x_1)}x_1^{m-a_{i,0}}P_{i,2}(x_1)\,dx_1. \tag{23}$$

Section III.

The Asymptotic Development of $y(x)$.

Taking $y(x)$ as given in (23), the discussion of its asymptotic development will fall into two parts—Case 1 and Case 2—according to the behavior of $f_i(x)$.

Case 1.

Suppose $f_i(x)\not\equiv 0$, $i=1,\ldots,n$. We shall order the f's so that*

$$R[f_1(x)]\geqq R[f_2(x)]\geqq R[f_3(x)]\geqq\ldots\geqq R[f_v(x)]>0$$
$$\geqq R[f_{v+1}(x)]\geqq R[f_{v+2}(x)]\geqq\ldots\geqq R[f_n(x)]$$

when x is large, real, and positive. Let β_i, $i=1,\ldots,n$ be so chosen that $\beta_1=\beta_2=\ldots=\beta_v=\infty$ and $\beta_{v+1}=\beta_{v+2}=\ldots=\beta_n=\beta$, where β is to be taken sufficiently large. Then $y(x)$ in (23) becomes

$$y(x)=\sum_{i=1}^{v}S_i(x)+\sum_{i=v+1}^{n}T_i(x), \tag{24}$$

where

$$S_i(x)=-\int_x^{\infty} e^{f_i(x)}x^{a_{i,0}-(n-1)k}P_{i,1}(x)\,e^{-f_i(x_1)}x_1^{m-a_{i,0}}P_{i,2}(x_1)\,dx_1,\quad i=1,\ldots,v,$$

$$T_i(x)=\int_{\beta}^{x} e^{f_i(x)}x^{a_{i,0}-(n-1)k}P_{i,1}(x)\,e^{-f_i(x_1)}x_1^{m-a_{i,0}}P_{i,2}(x_1)\,dx_1,\quad i=v+1,\ldots,n.$$

Let s_i be the dominant term in S_i, $i=1,\ldots,v$. It can be written in the form

$$s_i(x)=e^{f_i(x)}x^{a_{i,0}-(n-1)k}\int_x^{\infty}\frac{x_1^{m-a_{i,0}}}{f'(x_1)}[-f'(x_1)]\,dx_1,\qquad i=1,\ldots,v,$$

in which $f_i'(x_1)\not\equiv 0$, since $f_i(x_1)\not\equiv 0$ for x_1 sufficiently large. An integration by parts gives

$$\begin{aligned}s_i(x)=e^{f_i(x)}x^{a_{i,0}-(n-1)k}[\beta_{i,1,0}x_1^{m-a_{i,0}-k}\\
+\ldots+\beta_{i,1,p_{i,1}}x_1^{m-a_{i,0}-k-p_{i,1}}[1+\varepsilon_{i,1}(x_1)]]\,e^{-f_i(x_1)}]_x^{\infty}\\
+e^{f_i(x)}x^{a_{i,0}-(n-1)k}\int_x^{\infty}(\gamma_{i,1,0}x_1^{m-a_{i,0}-k-1}\\
+\ldots+\gamma_{i,1,p_{i,1}}x_1^{m-a_{i,0}-k-p_{i,1}}[1+\varepsilon_{i,2}(x_1)])\,e^{-f_i(x_1)}dx_1,\quad i=1,\ldots,v,\end{aligned}$$

where

$$\beta_{i,1,0}\ldots\beta_{i,1,p_{i,1}},\quad \gamma_{i,1,0}\ldots\gamma_{i,1,p_{i,1}}$$

* By $R(a)$ we shall mean the real part of a.

are easily determined constants, and

$$\lim_{x=\infty} \varepsilon_{i,1}(x) = \lim_{x=\infty} \varepsilon_{i,2}(x) = 0.$$

By repeated application of integration by parts the exponents of the x's throughout are continually decreased. After l applications of integration by parts, the part still affected by the integral sign, call it $I_i(x)$, is the following:

$$I_i(x) = e^{f_i(x)} x^{a_{i,0}-(n-1)k} \int_x^\infty (\gamma_{i,l,0} x_1^{m-a_{i,0}-l(k+1)} + \gamma_{i,l,1} x_1^{m-a_{i,0}-l(k+1)-1}$$
$$+ \ldots + \gamma_{i,l,p_{i,l}} x_1^{m-a_{i,0}-l(k+1)-p_{i,l}} [1+\varepsilon_{i,2l}(x_1)]) e^{-f_i(x_1)} dx_1, \quad i=1, \ldots, v,$$

where the γ's are easily determined constants and $\lim_{x=\infty} \varepsilon_{i,2l}(x) = 0$. Suppose l is chosen so large that $R(m-a_{i,0}-l(k+1)) \leqq -2$. Since $e^{-f_i(x)}$ $i=1, \ldots, v$ is a monotonic decreasing function of x, a positive quantity M_i may be found such that

$$|I_i(x)| \leqq |x^{m-nk}| \frac{M_i}{x^{(k+1)(l-1)}}. \tag{27}$$

For l greater than unity the last factor here appearing may be made small at pleasure with large values of x. By referring to (25), (26) and (27), it is readily seen that s_i and consequently S_i, $i=1, \ldots, v$ can be represented asymptotically in the form

$$S_i(x) \sim \left[A_{i,0} + \frac{A_{i,1}}{x} + \frac{A_{i,2}}{x^2} + \ldots\right]. \tag{28}$$

Let t_i be the dominant term in T_i, $i=v+1, \ldots, n$. An integration by parts gives

$$t_i = e^{f_i(x)} x^{a_{i,0}-(n-1)k} (\beta_{i,1,0} x_1^{m-a_{i,0}-k}$$
$$+ \ldots + \beta_{i,1,p_{i,1}} x_1^{m-a_{i,0}-k-p_{i,1}} [1+\varepsilon_{i,1}(x_1)]) e^{-f_i(x_1)}]_\beta^x$$
$$+ e^{f_i(x)} x^{a_{i,0}-(n-1)k} \int_\beta^x (\gamma_{i,1,0} x_1^{m-a_{i,0}-k-1}$$
$$+ \ldots + \gamma_{i,1,p_{i,1}} x_1^{m-a_{i,0}-k-p_{i,1}-1} [1+\varepsilon_{i,2}(x_1)]) e^{-f_i(x_1)} dx_1, \quad i=v+1, \ldots, n, \tag{29}$$

where $\beta_{i,1,0}, \ldots, \beta_{i,1,p_{i,1}}$ and $\gamma_{i,1,0}, \ldots, \gamma_{i,1,p_{i,1}}$ are easily determined constants, and,

$$\lim_{x=\infty} \varepsilon_{i,1}(x) = \lim_{x=\infty} \varepsilon_{i,2}(x) = 0.$$

After l applications of the integrations by parts, the integral that still remains is as follows:

$$I_i(x) = e^{f_i(x)} x^{a_{i,0}-(n-1)k} \int_\beta^x (\gamma_{i,l,0} x_1^{m-a_{i,0}-l(k+1)}$$
$$+ \ldots + \gamma_{i,l,p_{i,l}} x_1^{m-a_{i,0}-l(k+1)-p_{i,l}} [1+\varepsilon_{i,2l}(x_1)]) e^{-f_i(x_1)} dx_1,$$
$$i=v+1, \ldots, n. \tag{30}$$

For large values of β, and for $x>\beta$, each term in the integrand of (30) is a monotone increasing function of x_1 in the interval of integration. A positive quantity M_i can be found such that

$$|I_i(x)|\leq|x^{m-nk}|\frac{M_i}{x^{(l-1)(k+1)}}, \qquad i=v+1, \ldots, n. \tag{31}$$

For l greater than unity the last factor here appearing can be made small at pleasure for large values of x. From (29), (30) and (31) it follows that t_i, and consequently T_i, $i=v+1, \ldots, n$, can be developed asymptotically in the form

$$T_i(x)\sim\left[A_{i,0}+\frac{A_{i,1}}{x}+\frac{A_{i,2}}{x^2}+\ldots\right]. \tag{32}$$

Case 2.

Suppose $f_r(x)\equiv 0$ where r is one and only one of the integers $1, \ldots, n$. Only one f could be identically zero for the roots of the characteristic equation (16), are distinct.

Referring to (23), the part arising from $f_r(x)$, in the expression for $y(x)$, is as follows:

$$u(x)=x^{a_{i,0}-(n-1)k}\left(c_{r,0}+\frac{c_{r,1}}{x}+\ldots+\frac{c_{r,n}+\varepsilon_p(x)}{x^p}\right)$$
$$\int_\beta^x x_1^{m-a_{i,0}}\left(d_{r,0}+\frac{d_{r,1}}{x_1}+\ldots+\frac{d_{r,q}+\varepsilon_q(x_1)}{x_1^q}\right)dx_1,$$

where p and q are arbitrary positive integers and $\lim_{x=\infty}\varepsilon_p(x)=\lim_{x=\infty}\varepsilon_q(x)'=0$. It follows that

$$u(x)\sim x^{m-nk+k+1}\left[B_0+\frac{B_1}{x}+\frac{B_2}{x^2}+\ldots\right],$$

provided that $m-a_{r,0}\neq s$, when s takes on the values $-1, 0, 1, \ldots, q-1$. If, however, $m-a_{r,0}=s$, then the development of $u(x)$ is

$$u(x)\sim x^{m-nk+k+1}\left[B_0+\frac{B_1}{x}+\ldots\right]+x^{a_{r,0}-nk+k}\left[c_0+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots\right]\log x.$$

That part of the expansion of (23) arising from $i=1, \ldots, r-1, r+1, \ldots, n$ takes the same form as (28) or (32) of Case 1.

In summary we are able to state the following:

Theorem II. *In the differential equation*

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\ldots+a_n(x)y=X(x), \tag{1}$$

suppose that a_i and X are real or complex functions of x developable asymptotically, when x is large, real, and positive, in the forms

$$a_i(x) \sim x^{ik}\left[a_{i,0} + \frac{a_{i,1}}{x} + \frac{a_{i,2}}{x^2} + \dots\right], \quad i=1, \dots, n; \quad k=0, 1, 2, \dots.$$

$$X(x) \sim x^m\left[b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} \dots\dots\right], \quad b_0 \neq 0,$$

while the first $n-i$ derivatives of $a_i(x)$ also possess asymptotic developments.

Consider the roots $w_1 \dots w_n$ of the characteristic equation

$$w^n + a_{1,0}w^{n-1} + a_{2,0}w^{n-2} + \dots + a_{n,0} = 0$$

as all distinct. Let z_i represent certain determinate functions of x of the form

$$z_i(x) = e^{-f_i(x)}x^{-\alpha_{i,0}}P_i(x),$$

where

$$f_i(x) = \frac{w_i x^{k+1}}{k+1} + \alpha_{i,-k}\frac{x^k}{k} + \alpha_{i,-k+1}\frac{x^{k-1}}{k-1} + \dots + \alpha_{i,-1}x,$$

and

$$P_i(x) \sim \left[c_{i,0} + \frac{c_{i,1}}{x} + \frac{c_{i,2}}{x^2} + \dots\right]; \quad i=1, \dots, n,$$

when x is large, real, and positive. Then for the same values of x there exists a particular solution, $y(x)$, of (1), which can be developed asymptotically as follows:

(a) $y(x) \sim x^{m-nk}\left[A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \dots\right]$, *if* $f_i(x) \not\equiv 0$, $i=1, \dots, n$.

(b) $y(x) \sim x^{m-(n-1)k+1}\left[B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \dots\right]$, *if* $f_r(x) \equiv 0$, *where r is one and only one of the values 1, 2, ..., n, and if at the same time* $m - \alpha_{r,0} \neq S$, *when S takes on the values* $-1, 0, 1, 2 \dots$

(c) $y(x) \sim x^{m-(n-1)k+1}\left[B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \dots\right] + x^{\alpha_{r,0}-(n-1)k}\left[c_0 + \frac{c_1}{x} + \dots\right] \log x$,

if $f_r(x) \equiv 0$ where r is one and only one of the values 1, 2, ..., n, and if at the same time $m - \alpha_{r,0} = S$ when S is minus one, zero, or a positive integer.

UNIVERSITY OF MICHIGAN,

A Collineation Group Isomorphic with the Group of the Double Tangents of the Plane Quartic.

By C. C. Bramble.

Introduction.

The group of the double tangents of a plane quartic is isomorphic with one of a series of groups arising in connection with the theta functions. This one is associated with the division into half-periods for $p=3$. Its immediate predecessor associated analytically with the division into half-periods for $p=2$ is the group of order $16 \cdot 720$ associated geometrically with the Kummer* surface. A similar one† is determined by the division into thirds of periods for theta functions for $p=2$, and is associated geometrically with the lines of a cubic surface. In all these cases isomorphic collineation groups have been discovered and discussed in considerable detail, but no collineation group isomorphic with the group of the double tangents has been discussed. It is the purpose of this paper to derive such a group. The group being connected with the quartic curve, by proper mapping methods a collineation group is obtained in which the variables are irrational invariants of the quartic itself. The equation of the quartic and its double tangents are obtained in a form whose symmetry and simplicity leave nothing to be desired. A complete system for the collineation group and associated canonical forms of the quartic are obtained. The collineation group appears in seven variables. That this is the smallest number of variables in which this group can be represented as a collineation group is evident from a theorem of Wiman in Weber, "Lehrbuch der Algebra," Vol. II, p. 376. The results obtained are applicable to the solution of the equation of the double tangents of the quartic, and should also be valuable for discussing certain invariants of the quartic and configurations of the double tangents. The quartic appears with an isolated flex and may throw some light on the hitherto unsolved problem of the flexes.

* An account of this group in relation to the Kummer surface is to be found in Hudson's "Kummer's Quartic Surface," which appeared in 1905.

† This group was discussed by Burkhardt, who gave an historical account of the matter up to the time of his papers (about 1890). They appeared in the *Math. Annalen*, Vols. XXXV, XXXVI, XXXVIII.

I.

The Cremona Group $G_{7,2}$ of P_7^2 in S_6.

Two sets of seven points in a plane, P_7^2 and Q_7^2, ordered with respect to each other, are *congruent under the Cremona transformation* C_m with ρ F-points if ρ of the pairs p_i, q_i $(i=1, 2, \ldots, 7)$ are corresponding F-points of C_m, and if the remaining $7-\rho \geqq 0$ of the pairs p_i, q_i are pairs of ordinary corresponding points under C_m. The number of projectively distinct sets congruent to P_7^2 is the number of types of Cremona transformations. To determine this number the following theorem* is necessary:

The general Cremona transformation C_m $(m>2)$ with ρ F-points is projectively determined when there are given the order m, the ρ F-points, their multiplicities subject to the conditions $\sum_1^\rho r_i^2=m^2-1$, $\sum_1^\rho r_i=3(m-1)$, and the positions of four corresponding F-points.

The possible transformations to be considered in connection with P_7^2 are given by the following table where α_j is the number of F-points of multiplicity j:

	C_2	C_3	C_4	D_4	C_5	D_5	D_6	D_7	D_8
α_1	3	4	3	6	0	3	1	0	0
α_2		1	3	0	6	3	4	3	0
α_3				1		1	2	4	7

C is used to indicate a transformation with six or fewer F-points, D one with seven F-points. Using in addition the collineation C_1 we find the number of transformations C_1, C_2, C_3, C_4, C_5, D_4, D_5, D_6, D_7, D_8 to be respectively $\binom{7}{0}$, $\binom{7}{3}$, $\binom{7}{4}\binom{3}{1}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{6}$, $\binom{7}{6}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{1}\binom{6}{4}$, $\binom{7}{3}$, $\binom{7}{0}$ or 2.288. But since P_7^2 and Q_7^2 congruent under D_8 are projective, there are only 288 projectively distinct types of congruence.

The sets P_7^2 and Q_7^2 are mapped upon the points of a space S_6 by taking them in the canonical form:

$$P_7^2\colon\ (1,0,0),\quad (0,1,0),\quad (0,0,1),\quad (1,1,1),\quad (x_i, y_i, u),\quad (i=1,2,3),$$
$$Q_7^2\colon\ (1,0,0),\quad (0,1,0),\quad (0,0,1),\quad (1,1,1),\quad (x'_j, y'_j, u),\quad (j=1,2,3),$$

and regarding $x_1, x_2, x_3, y_1, y_2, y_3, u$ as the coordinates of a point in S_6. Then, if two sets of points are congruent under a Cremona transformation in S_2, their maps in S_6 are corresponding points under a Cremona transformation

* Coble, "Point Sets and Allied Cremona Groups," Part II, *Transactions of the American Mathematical Society*, Vol. XVII, p. 348.

in S_6. The general Cremona transformation in S_2 can be expressed as a product of quadratic factors. The effect in S_6 corresponding to a quadratic transformation in S_2 is that of an involutory Cremona transformation. Moreover, any Cremona transformation of the kind considered is a product of transformations corresponding to quadratic transformations in S_2. Any quadratic transformation in S_2 with F-points at points of P_7^2 can be obtained from a single one by permutation of the points. Hence $G_{7,2}$, the Cremona group of P_7^2 in S_6, can be generated by the symmetric group of order 7! and a single transformation in S_6 corresponding to a quadratic transformation in S_2. The number of operations in $G_{7,2}$ is clearly the same as the number of types of congruence of sets P_7^2 if further it is required that P_7^2 be ordered. $G_{7,2}$ is thus seen to be of order 7!288.

II.

Point Sets on a Cuspidal Cubic.

The cuspidal cubic curve $C_1 \equiv x_2^3 - x_1 x_3^2 = 0$ is given parametrically by $x_1 = t^3$; $x_2 = t$; $x_3 = 1$, the parameter of the cusp being $t = \infty$ and that of the flex being $t = 0$. Hence given C_1, a set of seven points P_7^2 is determined by seven parameters t_i $(i = 1, 2, \ldots, 7)$. If, on the other hand, seven points only are given, they determine a net of cubics containing among them twenty-four cuspidal cubics. Thus the fact that C_1 is given is equivalent to the assumption of a single solution of the cusp equation of degree 24 of the net. P_7^2 determined in this way is general.

The condition that two points coincide is

$$t_i - t_j = 0. \tag{1}$$

The condition that three points t_i be on a line is

$$\sum^{3} t_i = 0. \tag{2}$$

The condition that six points t_i be on a conic is

$$\sum^{6} t_i = 0. \tag{3}$$

The quadratic transformation A_{123} with F-points at t_1, t_2 and t_3 sends C_1 into another cuspidal cubic C_1' whose points can be named by means of the same parameter t. C_1' can be sent by a collineation into C_1. This operation sends a point t on C_1' into a point t' of C_1. To determine the effect on the parameters we note that if

$$t_i' = t_i + \tfrac{1}{3}(t_1 + t_2 + t_3), \tag{4}$$

then

$$t_i' + t_j' + t_k' = t_i + t_j + t_k + t_1 + t_2 + t_3.$$

That is, the requirement that to three points on a line correspond three points on a conic through the F-points is satisfied. This gives the effect of the transformation on an ordinary point. It is clear that the condition that a point coincide with an F-point goes into the condition that the corresponding point be on the opposite F-line. Hence

$$t'_i - t'_3 = t_i + t_1 + t_2,$$

and by means of (4) we obtain the relation

$$t'_3 = t_3 - \tfrac{2}{3}(t_1 + t_2 + t_3).$$

The effect of A_{123} is then that of the collineation on the parameters given by the equations

$$\begin{aligned} t'_1 &= t_1 - \tfrac{2}{3}(t_1+t_2+t_3), & t'_2 &= t_2 - \tfrac{2}{3}(t_1+t_2+t_3), \\ t'_3 &= t_3 - \tfrac{2}{3}(t_1+t_2+t_3), & t'_i &= t_i + \tfrac{1}{3}(t_1+t_2+t_3), \end{aligned} \qquad i=4, 5, 6, 7,$$

where t_1, t_2 and t_3 are F-points, and t_i ordinary points.

The aggregate of operations obtained by taking products of A_{123} and permutations of $t_1, t_2, \ldots, t_7$ constitute the group $T_{7,2}$ of P_7^2 on C_1. An element of $T_{7,2}$ can be looked upon as the operation of passing from one P_7^2 on C_1 to a congruent one named by seven other values t'_i. We get in this way 288 projectively distinct sets of points on C_1 congruent in some order. Hence there are $7!\,288$ projectively distinct ordered sets. $T_{7,2}$ the collineation group on the variables $t_1, \ldots, t_7$ is of order $7!\,288$.

Only the ratios of the t's are essential since the transformation $t'_i = \mu t_i$ represents a projectivity of C_1 into itself and therefore the sets $t_1, \ldots, t_7$ and $\mu t_1, \ldots, \mu t_7$ are projective.

III.

Invariants of $T_{7,2}$.

An invariant of $T_{7,2}$ is a function of the t's unaltered by the operations of $T_{7,2}$. The condition that two points coincide is sent by A_{123} into the condition that two points coincide, or that three points be on a line; the condition that three points be on a line is sent into the condition that two points coincide, that three points be on a line, or that six points be on a conic; the condition that six points be on a conic is sent into the condition that six points be on a conic or that three points be on a line. The algebraic expressions for these conditions ((1), (2) and (3) of II) are permuted as stated, but may change sign. Hence

$$I_2 = \sum_{21}(t_1-t_2)^2 + \sum_{35}(t_1+t_2+t_3)^2 + \sum_{7}(t_1+t_2+t_3+t_4+t_5+t_6)^2 = 9(3a_1^2 - 4a_2),$$

where a_i is a symmetric function of the t's of degree i, is an invariant of $T_{7,2}$ of degree 2. Likewise are found,

$$I_4 = \sum_{21}(t_1-t_2)^4+\sum_{35}(t_1+t_2+t_3)^4+\sum_{7}(a_1-t_1)^4 = 3(3a_1^2-4a_2)^2,$$
$$\begin{aligned} I_6 &= \sum_{21}(t_1-t_2)^6+\sum_{35}(t_1+t_2+t_3)^6+\sum_{7}(a_1-t_1)^6 \\ &= 27a_1^6-108a_1^4a_2+192a_1^2a_2^2-96a_2^3-72a_1^3a_3+24a_1a_2a_3-36a_3^2+72a_1^2a_4 \\ &\quad +48a_2a_4-72a_1a_5-288a_6, \end{aligned}$$

which are invariants of degrees 4 and 6, respectively. I_4 is seen to be a multiple of I_2^2.

IV.

The Quartic C^4 Arising from a Set of Seven Points.

The plane E_x of P_7^2 is mapped upon a plane E_y by the cubic curves on P_7^2. To the cubics of E_x correspond the lines of E_y. Hence to two residual base points of a pencil of cubics of the net on P_7^2 there corresponds one point of E_y. If, however, a double point of a curve of the net on P_7^2 is taken, it alone corresponds to a point of E_y, for the two variable intersections of curves of the net have coincided. Since the Jacobian is the locus of double points of the net, the correspondence between the Jacobian of the net on P_7^2 and its map in E_y is one to one. The Jacobian of the net on P_7^2 has double points at the points of P_7^2, and being of order 6, will have $6\times3-7\times2=4$ variable intersections with curves of the net. That is, the map of the Jacobian squared, since pairs of points have coincided on it, is a quartic curve C^4 in E_y. The cubics of the net with a double point map into the lines of C^4, but the twenty-one degenerate cubics P_{ij}, consisting of the line t_it_j and the conic on the remaining five points, and the seven cubics P_{0i}, with a double point at a point of P_7^2, map into the twenty-eight double tangents of C^4 in such a way that the seven P_{0i} give rise to an Aronhold set. The twenty-four cuspidal cubics of the net map into the flex tangents and the twenty-four cusps into the flexes of C^4.

The operations of $T_{7,2}$ transform P_7^2 into Q_7^2, and transform the net of cubics on P_7^2 into a net on Q_7^2. The curves P_{0i} and P_{ij} of the net on P_7^2 are transformed into the curves Q_{0i} and Q_{ij} of Q_7^2. The effect of the generating transformation A_{123} on the curves P_{0i} and P_{ij} is to send them into Q_{0i} and Q_{ij} in such a way as to bring about the following permutation of the pairs of subscripts:

$$(01, 23)\,(02, 31)\,(03, 12)\,(45, 67)\,(46, 57)\,(47, 56),$$

the other pairs being unaltered.

Hence, the effect of the product $A_{237}A_{457}A_{167}$ is that of the interchange of subscripts 0 and 7. $G_{7!}$ together with the transposition (07) generates a subgroup of $T_{7,2}$, the symmetric group $G_{8!}$ of the permutations of $0, 1, 2, \ldots, 7$. By comparing the notation above for the curves P with the Hesse notation of the double tangents $[ik;\ i, k=1, \ldots, 8;\ i \neq k]$ of the quartic, it is seen at once that $G_{8!}$ and A_{123} effect the same permutations on the curves P as the subgroup* E and the substitution P_{1238}, which generate the group of the double tangents, effect on the double tangents. Since the order of $T_{7,2}$ is that of the group of the double tangents of the quartic,

$T_{7,2}$ is simply isomorphic with the group of the double tangents of the quartic.

V.

A Net of Cubics on P_7^2.

We will obtain the quartic map of the Jacobian for the net of cubics on $t_1, t_2, \ldots, t_7$ formed by taking the three following base cubics

$C_1 \equiv -x_1x_3^2+x_2^3=0$, the cuspidal cubic above;

$$C_2 \equiv x_1^2x_2-a_1x_1^2x_3+a_2x_1x_2^2-a_3x_1x_2x_3+a_4x_2^3-a_5x_2^2x_3+a_6x_2x_3^2-a_7x_3^3=0,$$

the cubic on $t_1, \ldots, t_7$ having no term in $x_1x_3^2$ and passing through the cusp of C_1;

$$C_3 \equiv 4x_1^3+x_1^2(c_2x_2-c_3x_3)+x_1(c_4x_2^2-c_5x_2x_3)+c_6x_2^3-c_7x_2^2x_3+c_8x_2x_3^2-c_9x_3^3=0,$$

the cubic on $t_1, \ldots, t_7$ having no term in $x_1x_3^2$ and touching C_1 at $t=-\frac{1}{2}a_1$. The a_i are symmetric functions of $t_1, \ldots, t_7$ of degree i; the c's are given in terms of the a's by

$$c_n=4a_n-4a_1a_{n-1}+a_1^2a_{n-2}.$$

VI.

The Jacobian $J[C_1C_2C_3]$ and its Map C^4.

We have now to determine the Jacobian J of C_1, C_2, and C_3, and obtain the quartic map of J^2 by means of the equations

$$x_1'=C_1, \quad x_2'=C_2, \quad x_3'=C_3.$$

$$\begin{aligned}
J \equiv\ & 24x_1^5x_3-36a_1x_1^4x_2^2+48a_2x_1^4x_2x_3+(-24a_3+3a_1c_2-3c_3)x_1^4x_3^2 \\
& +(-36a_3-6a_1c_2+6c_3)x_1^3x_2^3+(72a_4+6a_2c_2-6c_4)x_1^3x_2^2x_3 \\
& +(-48a_5-3a_3c_2-6a_2c_3+6a_1c_4+3c_5)x_1^3x_2x_3^2+(24a_6+3a_3c_3-3a_1c_5)x_1^3x_3^3 \\
& +(-36a_5-6a_3c_2+3a_2c_3-3a_1c_4+6c_5)x_1^2x_2^4 \\
& +(72a_6+12a_4c_2+3a_3c_3-3a_1c_5-12c_6)x_1^2x_2^3x_3
\end{aligned}$$

---

* "Finite Groups," Miller, Blichfeldt and Dickson, pp, 362–365.

$$+(-108a_7-9a_5c_2-9a_4c_3+9a_1c_6+9c_7)x_1^2x_2^2x_3^2+(6a_6c_2+6a_5c_3-6a_1c_7-6c_8)x_1^2x_2x_3^3$$
$$+(-3a_7c_2-3a_6c_3+3a_1c_8+3c_9)x_1^2x_3^4+(-6a_5c_2-3a_3c_4+3a_2c_5+6c_7)x_1x_2^5$$
$$+(12a_6c_2+6a_5c_3+6a_4c_4-6a_2c_6-6a_1c_7-12c_8)x_1x_2^4x_3$$
$$+(-18a_7c_2-12a_6c_3-6a_5c_4-3a_4c_5+3a_3c_6+6a_2c_7+12a_1c_8+18c_9)x_1x_2^3x_3^2$$
$$+(18a_7c_3+6a_6c_4+3a_5c_5-3a_3c_7-6a_2c_8-18a_1c_9)x_1x_2^2x_3^3$$
$$+(-6a_7c_4-3a_6c_5+3a_3c_8+6a_2c_9)x_1x_2x_3^4+(3a_7c_5-3a_3c_9)x_1x_3^5$$
$$+(-3a_5c_4+3a_2c_7)x_2^6+(3a_6c_4+3a_5c_5-3a_3c_7-6a_2c_8)x_2^5x_3$$
$$+(-9a_7c_4-6a_6c_5-3a_5c_6+3a_4c_7+6a_3c_8+9a_2c_9)x_2^4x_3^2$$
$$+(9a_7c_5+6a_6c_6-6a_4c_8-9a_3c_9)x_2^3x_3^3+(-9a_7c_6-3a_6c_7+3a_5c_8+9a_4c_9)x_2^2x_3^4$$
$$+(6a_7c_7-6a_5c_9)x_1x_3^5+(-3a_7c_8+3a_6c_9)x_3^6=0.$$

$\left(\frac{J}{3}\right)^2$ expressed in terms of the cubics C_1, C_2, and C_3, *i. e.*, the map C^4 in E_y if the C's are regarded as reference lines is:

$$\left(\frac{J}{3}\right)^2\equiv(64a_1a_2a_3a_5a_7-32a_2a_3a_6a_7-64a_1a_3^2a_4a_7+32a_1a_2^2a_6a_7$$
$$-64a_1^2a_2^2a_5a_7+64a_1^2a_2a_3a_4a_7-64a_4a_7^2-64a_1a_5^2a_7+128a_1^2a_4a_5a_7$$
$$-64a_1^3a_4^2a_7+64a_5a_6a_7-64a_1a_4a_6a_7+16a_2^2a_7^2+16a_3^2a_6^2+16a_1^2a_2^2a_6^2$$
$$-32a_1a_2a_3a_6^2)C_1^4+(8a_1^2a_3^2a_6-8a_1^3a_2a_3a_6+80a_1^3a_4a_7-72a_1^2a_2a_3a_7$$
$$+16a_1^3a_2^2a_7+16a_1^2a_6^2-16a_1^3a_5a_6+16a_1^4a_4a_6-112a_1^2a_5a_7+64a_1a_3^2a_7$$
$$+64a_3a_4a_7-32a_1a_2a_4a_7+64a_7^2+64a_1a_6a_7+32a_1a_3a_4a_6-32a_1^2a_2a_4a_6$$
$$-32a_2a_5a_7-32a_3a_5a_6+32a_1a_2a_5a_6)C_1^3C_2+(8a_1a_2a_3a_6-8a_3^2a_6$$
$$+48a_1a_4a_7+8a_2a_3a_7-16a_1a_2^2a_7-16a_6^2+16a_1a_5a_6-16a_1^2a_4a_6-16a_5a_7)C_1^3C_3$$
$$+(16a_2a_6-16a_1a_2a_5+8a_1^3a_5-2a_1^2a_3^2+8a_1a_3a_4+8a_3a_5-48a_1a_7)C_1^2C_2C_3$$
$$+(a_1^4a_3^2-4a_1^5a_5-8a_1^4a_6+16a_1^2a_2a_6+16a_1^3a_2a_5-8a_1^3a_3a_4-8a_1^2a_3a_5$$
$$-16a_1^3a_7+32a_1a_2a_7-32a_1a_3a_6+16a_5^2-32a_1a_4a_5+16a_1^2a_4^2-64a_3a_7)C_1^2C_2^2$$
$$+(a_3^2-4a_1a_5+8a_6)C_1^2C_3^2+(a_1^6-4a_1^4a_2+8a_1^3a_3-16a_1^2a_4+32a_1a_5)C_1C_2^3$$
$$+(8a_1^2a_2-3a_1^4-8a_1a_3-16a_4)C_1C_2^2C_3+(3a_1^2-4a_2)C_1C_2C_3^2-C_1C_3^3+16C_2^3C_3=0.$$

The quartic C^4 is only projectively determined since the net of cubics is only projectively determined by the choice of seven points, that is, C_1, C_2, and C_3 can be any linearly independent cubics of the net. Moreover, any transformation sending the net of cubics into a net of cubics transforms C^4 point by point into itself. Suppose such a transformation is given by the equations $x_i=q_i(x')$. Then $C_i(x)$ goes into $C_i[q_1(x')\ldots.]\equiv C_i'(x')$. A point of C^4 is given by $y_i=C_i(x)$ where x is a point of J. But the transform of the point is $y_i'=C_i'(x')$. Since $C_i'(x')\equiv C_i[q_1(x')\ldots.]\equiv C_i(x)$, $y_i'=y_i$ and the point is unaltered. The curves P_{0i} and P_{ij} have, however, been permuted and the coefficients are those derived from the transformed Aronhold set and are in general altered in form. For, if the transformation A_{123} is applied to the set

of points $t_1, \ldots, t_7$, and to the cubics C_1, C_2, C_3 we obtain a new set of points $t'_1, \ldots, t'_7$ and three new cubics C'_1, C'_2, C'_3 whose coefficients contain not only the symmetric functions of the points $t'_1, \ldots, t'_7$, but in addition the F-points t'_1, t'_2, and t'_3. Likewise the transform of C_4 by A_{123}, that is, the map of $J^2[C'_1, C'_2, C'_3]$ by C'_1, C'_2, and C'_3 will contain the F-points t'_1, t'_2, and t'_3, besides the symmetric functions of $t'_1, \ldots, t'_7$. If, however, C_1, C_2, and C_3 were cubics covariant under A_{123}, then since J also is covariant, the coefficients of the quartic C^4 would be invariants of A_{123}, and since they are symmetric would be invariants of $T_{7,2}$. The cubics C_1, C_2, and C_3 are mapped into the reference lines of the plane E_y. Since C_1 is covariant and is mapped into the known flex tangent, a triangle of reference determined uniquely by the flex would arise from a set of cubics covariant with the cusp cubic. The problem of finding the above-mentioned invariants of $T_{7,2}$ is then reduced to that of finding the coefficients of C^4 referred to a triangle of reference covariant with the flex.

A simple way to determine such a covariant triangle is to take

(1) for x_1 the line C_1, the flex tangent;

(2) for x_3 the tangent to C^4 at the intersection of C_1 with C^4 other than the flex $(0, 0, 1)$;

(3) for x_2 the line joining $0, 0, 1$ with the intersection of x_3 with the polar line of $0, 0, 1$ as to the polar conic of $0, 1, 0$ as to C^4.

The above choice of reference lines gives the following linear transformation of the C's to the new variables x:

$$C_1 = x_1, \quad C_2 = \lambda x_1 + x_2, \quad C_3 = \mu x_1 + x_3,$$

where $48\lambda = 3a_1^4 - 8a_1^2a_2 + 8a_1a_3 + 16a_4$ and $16\mu = -a_1^6 + 4a_1^4a_2 - 8a_1^3a_3 + 16a_1^2a_4 - 32a_1a_5$.

The following expressions A_i, invariants of degree i of $T_{7,2}$, are such numerical multiples of the coefficients of the transform of C^4 as to remove fractional coefficients. α_{ijkl} is the coefficient of $x_ix_jx_kx_l$ where i, j, k, l are 1, 2, or 3.

$$A_2 = \alpha_{1233} = 3a_1^2 - 4a_2.$$

$$A_6 = 48\alpha_{1133} = 18a_1^6 - 72a_1^4a_2 + 96a_1^3a_3 + 32a_1^2a_2^2 - 96a_1^2a_4 - 32a_1a_2a_3 + 96a_1a_5 - 64a_2a_4 + 48a_3^2 + 384a_6.$$

$$A_8 = 48\alpha_{1123} = -27a_1^8 + 144a_1^6a_2 - 192a_1^5a_3 - 160a_1^4a_2^2 + 192a_1^4a_4 + 320a_1^3a_2a_3 - 192a_1^3a_5 - 128a_1^2a_2a_4 - 160a_1^2a_3^2 + 128a_1a_3a_4 - 2304a_1a_7 + 768a_2a_6 + 384a_3a_5 - 256a_4^2.$$

$$A_{10} = 16\alpha_{1122} = 3a_1^{10} - 20a_1^8a_2 + 32a_1^7a_3 + 32a_1^6a_2^2 - 32a_1^6a_4 - 96a_1^5a_2a_3 + 32a_1^5a_5 + 64a_1^4a_2a_4 + 80a_1^4a_3^2 - 128a_1^4a_6 - 128a_1^3a_3a_4 - 256a_1^3a_7 + 256a_1^2a_2a_6 + 128a_1^2a_3a_5 + 512a_1a_2a_7 - 512a_1a_3a_6 - 1024a_3a_7 + 256a_5^2.$$

$$\begin{aligned}
A_{12}=6912\alpha_{1113}=&-117a_1^{12}+936a_1^{10}a_2-1152a_1^9a_3-2352a_1^8a_2^2+5376a_1^7a_2a_3+4608a_1^7a_5\\
&+1536a_1^6a_2a_4-2816a_1^6a_3^2-6912a_1^6a_6-5376a_1^5a_2^2a_3-8064a_1^5a_2a_5+1152a_1^5a_3a_4\\
&-20736a_1^5a_7-4608a_1^4a_2^2a_4+8064a_1^4a_2a_3^2+34560a_1^4a_2a_6+23040a_1^4a_3a_5-2304a_1^4a_4^2\\
&+12288a_1^3a_2^2a_5+2304a_1^3a_2a_3^2+3072a_1^3a_2a_3a_4+55296a_1^3a_2a_7-10240a_1^3a_3^3\\
&-55296a_1^3a_3a_6-36864a_1^3a_4a_5-18432a_1^2a_2^2a_6-21504a_1^2a_2a_3a_5+6144a_1^2a_2a_4^2\\
&+12288a_1^2a_3^2a_4-55296a_1^2a_3a_7+101376a_1^2a_5^2-110592a_1a_2^2a_7+73728a_1a_2a_3a_6\\
&-24576a_1a_2a_4a_5-18432a_1a_3^2a_5+6144a_1a_3a_4^2+22184a_1a_4a_7-110592a_1a_5a_6\\
&+55296a_2a_3a_7+36864a_2a_4a_6-55296a_3^2a_6+13824a_3a_4a_5-8192a_4^3-110592a_5a_7\\
&-110592a_6^2.
\end{aligned}$$

$$\begin{aligned}
A_{14}=768\alpha_{1112}=&\,27a_1^{14}-252a_1^{12}a_2+384a_1^{11}a_3+752a_1^{10}a_2^2-384a_1^{10}a_4-2176a_1^9a_2a_3\\
&+384a_1^9a_5-608a_1^8a_2^3+1792a_1^8a_2a_4+1568a_1^8a_3^2-768a_1^8a_6+2816a_1^7a_2^2a_3\\
&-1280a_1^7a_2a_5-2944a_1^7a_3a_4+768a_1^7a_7-1536a_1^6a_2^2a_4-3968a_1^6a_2a_3^2+2816a_1^6a_2a_6\\
&+2432a_1^6a_3a_5+1280a_1^6a_4^2+4608a_1^5a_2a_3a_4-2048a_1^5a_2a_7+2048a_1^5a_3^3-5120a_1^5a_3a_6\\
&-2048a_1^5a_4a_5-1024a_1^4a_2^2a_6-512a_1^4a_2a_3a_5-1024a_1^4a_2a_4^2-4096a_1^4a_3^2a_4\\
&+8192a_1^4a_3a_7+8192a_1^4a_4a_6-1536a_1^4a_5^2+4096a_1^3a_2^2a_7+2048a_1^3a_3^2a_5+2048a_1^3a_3a_4^2\\
&+16384a_1^3a_4a_7-12288a_1^3a_5a_6-30720a_1^2a_2a_3a_7-4096a_1^2a_2a_4a_6+8192a_1^2a_2a_5^2\\
&-2048a_1^2a_3^2a_6-2048a_1^2a_3a_4a_5-12288a_1^2a_5a_7+12288a_1^2a_6^2-8192a_1a_2a_4a_7\\
&+32768a_1a_3^2a_7+8192a_1a_3a_4a_6-8192a_1a_3a_5^2+49152a_1a_6a_7-24576a_2a_5a_7\\
&+16384a_3a_4a_7-24576a_3a_5a_6+8192a_4a_5^2+49152a_7^2.
\end{aligned}$$

$$\begin{aligned}
A_{18}=9\cdot\overline{16}^3\alpha_{1111}=&\,63a_1^{18}-756a_1^{16}a_2+1152a_1^{15}a_3+3264a_1^{14}a_2^2-1152a_1^{14}a_4-9600a_1^{13}a_2a_3\\
&+1728a_1^{13}a_5-5600a_1^{12}a_2^3+8448a_1^{12}a_2a_4+7056a_1^{12}a_3^2-1152a_1^{12}a_6+24576a_1^{11}a_2^2a_3\\
&-12288a_1^{11}a_2a_5-13824a_1^{11}a_3a_4+4608a_1^{11}a_7+2816a_1^{10}a_2^4-16896a_1^{10}a_2^2a_4\\
&-34944a_1^{10}a_2a_3^2+6144a_1^{10}a_2a_6+21504a_1^{10}a_3a_5+6912a_1^{10}a_4^2-17152a_1^9a_2^3a_3\\
&+22016a_1^9a_2^2a_5+53760a_1^9a_2a_3a_4-29184a_1^9a_2a_7+17408a_1^9a_3^3-10752a_1^9a_3a_6\\
&-30720a_1^9a_4a_5+6656a_1^8a_2^3a_4+34816a_1^8a_2^2a_3^2-5120a_1^8a_2^2a_6-76288a_1^8a_2a_3a_5\\
&-21504a_1^8a_2a_4^2-46080a_1^8a_3^2a_4+52224a_1^8a_3a_7+61440a_1^8a_4a_6+20736a_1^8a_5^2\\
&-20480a_1^7a_2^2a_3a_4+106496a_1^7a_2^2a_7-31744a_1^7a_2a_3^3-8192a_1^7a_2a_3a_6+114688a_1^7a_2a_4a_5\\
&+72704a_1^7a_3^2a_5+36864a_1^7a_3a_4^2-24576a_1^7a_4a_7-73728a_1^7a_5a_6-8192a_1^6a_2^3a_6\\
&-4096a_1^6a_2^2a_3a_5-8192a_1^6a_2^2a_4^2+30720a_1^6a_2a_3^2a_4-299008a_1^6a_2a_3a_7\\
&-237568a_1^6a_2a_4a_6-61440a_1^6a_2a_5^2+11264a_1^6a_3^4+4096a_1^6a_3^2a_6-225280a_1^6a_3a_4a_5\\
&-8192a_1^6a_4^3+73728a_1^6a_6^2-212992a_1^5a_3^3a_7+229376a_1^5a_3^2a_7-4096a_1^5a_2a_3^2a_5\\
&+24576a_1^5a_2a_3a_4^2+139264a_1^5a_2a_4a_7+245760a_1^5a_2a_5a_6-12288a_1^5a_3^3a_4\\
&+229376a_1^5a_3^2a_7+385024a_1^5a_3a_4a_6+172032a_1^5a_3a_5^2+180224a_1^5a_4^2a_5\\
&+147456a_1^5a_6a_7+778240a_1^4a_2^2a_3a_7+81920a_1^4a_2^2a_4a_6-81920a_1^4a_2^2a_5^2\\
&-286720a_1^4a_2a_3^2a_6+40960a_1^4a_2a_3a_4a_5-32768a_1^4a_2a_4^3-122880a_1^4a_2a_5a_7\\
&-245760a_1^4a_2a_6^2+8192a_1^4a_3^3a_5-36864a_1^4a_3^2a_4^2-409600a_1^4a_3a_4a_7-466944a_1^4a_3a_5a_6\\
&-425984a_1^4a_4^2a_6-466944a_1^4a_4a_5^2+147456a_1^4a_7^2-327680a_1^3a_2^2a_4a_7-819200a_1^3a_2a_3^2a_7\\
&+163840a_1^3a_2a_3a_5^2-393216a_1^3a_2a_6a_7+163840a_1^3a_3^3a_6-16384a_1^3a_3^2a_4a_5
\end{aligned}$$

$$
\begin{aligned}
&+65536a_1^3a_3a_4^3+196608a_1^3a_3a_5a_7+393216a_1^3a_3a_6^2-262144a_1^3a_4^2a_7\\
&+1572864a_1^3a_4a_5a_6+294912a_1^3a_5^3-983040a_1^2a_2^2a_5a_7+589824a_1^2a_2^2a_6^2\\
&+1572864a_1^2a_2a_3a_4a_7-393216a_1^2a_2a_3a_5a_6-131072a_1^2a_2a_4^2a_6+131072a_1^2a_2a_4a_5^2\\
&-393216a_1^2a_2a_7^2+327680a_1^2a_3^3a_7-131072a_1^2a_3^2a_4a_6-81920a_1^2a_3^2a_5^2\\
&-65536a_1^2a_3a_4^2a_5+393216a_1^2a_3a_6a_7+393216a_1^2a_4a_5a_7-393216a_1^2a_4a_6^2\\
&-1179648a_1^2a_5^2a_6+1179648a_1a_2^2a_6a_7-1179648a_1a_2a_3a_6^2-262144a_1a_2a_4^2a_7\\
&+393216a_1a_3^2a_5a_6+262144a_1a_3a_4^2a_6-131072a_1a_3a_4a_5^2+393216a_1a_3a_7^2\\
&-1572864a_1a_4a_6a_7-1179648a_1a_5^2a_7+1179648a_1a_5a_6^2+589824a_2^2a_7^2\\
&-1179648a_2a_3a_6a_7-393216a_2a_4a_5a_7+589824a_3^2a_6^2+524288a_3a_4^2a_7\\
&-393216a_3a_4a_5a_6+65536a_4^2a_5^2-1572864a_4a_7^2+2359296a_5a_6a_7\,.
\end{aligned}
$$

The equation of C^4 referred to the chosen reference lines covariant with the flex is

$$
\begin{aligned}
3A_{18}x_1^4&+144A_{14}x_1^3x_2+16A_{12}x_1^3x_3+6912A_{10}x_1^2x_2^2+2304A_8x_1^2x_2x_3\\
&+2304A_6x_1^2x_3^2+110592A_2x_1x_2x_3^2-110592x_1x_3^3+1769472x_2^3x_3=0.
\end{aligned}
$$

VII.

Double Tangents of C^4.

The double tangents of C^4 in E_y are of two types:

(1) The type $(0i)$ are the maps of the cubic curves P_{0i} of the net in E_x with a double point at t_i. There are seven of this type and they form an Aronhold set.

(2) The type (ij) are the maps of the cubic curves P_{ij} of the net in E_x consisting of the line t_it_j and the conic on the remaining five points. There are twenty-one of this type.

To determine the equation of the double tangent $(0i)$ the equation of P_{0i} must first be found. The curves of the net having a common tangent at t_i form a pencil whose equation is

$$
\begin{aligned}
k(x_2^3-x_1x_3^2)+x_1^3&+(a_2-a_1^2-a_1t_i-t_i^2)x_1^2x_2-(a_3-a_1a_2-a_1^2t_i-a_1t_i^2)x_1^2x_3\\
&+(a_4-a_1a_3-a_1a_2t_i-a_2t_i^2)x_1x_2^2-(a_5-a_1a_4-a_1a_3t_i-a_3t_i^2)x_1x_2x_3\\
&+(a_6-a_1a_5-a_1a_4t_i-a_4t_i^2)x_1x_3^2-(a_7-a_1a_6-a_1a_5t_i-a_5t_i^2)x_2^2x_3\\
&+(-a_1a_7-a_1a_6t_i-a_6t_i^2)x_2x_3^2-(-a_1a_7t_i-a_7t_i^2)x_3^3=0.
\end{aligned}
$$

There is a single member of this pencil with a double point at t_i. This is the curve for which k has such a value that the point t_i, when substituted in a derivative with respect to x_1 makes it vanish. This value of k is

$$k=t_i^6+a_2t_i^4+(a_1a_2-a_3)t_i^3-a_5t_i+(a_6-a_1a_5).$$

The map in E_y of the curve of the above pencil for this value of k, that is, the equation of the double tangent $(0i)$ is

$$4[t_i^6+a_2t_i^4+(a_1a_2-a_3)t_i^3-a_5t_i+(a_6-a_1a_5)]C_1-(a_1+2t_1)^2C_2+C_3=0.$$

The equation of the line t_it_j is

$$x_1+(s_2-s_1^2)x_2+s_1s_2x_3=0,$$

where s_k is the symmetric function of t_i and t_j of degree k. The equation of the conic on the remaining five points is

$$x_1^2+(\sigma_4-\sigma_1\sigma_3)x_2^2-\sigma_1\sigma_5x_3^2+(\sigma_2-\sigma_1^2)x_1x_2-(\sigma_3-\sigma_1\sigma_2)x_1x_3-(\sigma_5-\sigma_1\sigma_4)x_2x_3=0,$$

where σ_k is the symmetric function of the five points other than t_i and t_j. To obtain the equation of the double tangent (ij) in E_y we have to find the map of the product of the equations of the line and conic above. This product expressed in terms of C_1, C_2, and C_3, that is, the equation of the double tangent (ij) is, after removing numerical fractions

$$4(\sigma_1\sigma_5+s_1s_2\sigma_3-s_1s_2\sigma_1\sigma_2)C_1-(a_1-2s_1)^2C_2+C_3=0.$$

VIII.

Proof of the Completeness of the System of Invariants.

The determination of the t's depends on the separation of the double tangents of the quartic and the isolation of a single flex. The t's are then projective irrational invariants of the quartic. Any function of the t's of proper weight is therefore a projective invariant of the quartic. Hence any invariant of $T_{7,2}$ is an irrational invariant of the quartic, such that the only irrationality present is that of the flex. Such an invariant is expressible rationally and integrally in terms of the coefficients of the quartic and of the coordinates of the isolated flex. But since the quartic has for coefficients the invariants A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, and A_{18}, and the coordinates of the flex are $0, 0, 1$, every invariant of $T_{7,2}$ is rationally and integrally expressible in terms of the invariants A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, and A_{18}. Moreover, it is obvious when special values are given to $a_1, a_2, \ldots, a_7$ that no one of the invariants A_i can be expressed rationally and integrally in terms of the others. Hence none of them is superfluous and

The invariants A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, and A_{18} form a complete system for the group $T_{7,2}$.

IX.

The Jacobian of $A_2, A_4, \ldots, A_{18}$.

If an expression is alternating under operations of $T_{7,2}$ it contains as a factor $t_i - t_j$ and all its conjugate values under the operations of $T_{7,2}$. Hence it has as a factor

$$J \equiv \prod^{21}(t_i - t_j)\prod^{35}(t_i + t_j + t_k)\prod^{7}(a_1 - t_i),$$

which is an alternating expression of degree 63.

The Jacobian of $A_2, A_4, \ldots, A_{18}$ is an alternating expression of degree 63 and is therefore to within a numerical factor the product J. Since J at most changes sign when the operations of $T_{7,2}$ are carried out, its square is an invariant of degree 126 and is rationally expressible in terms of the invariants $A_2, A_4, \ldots, A_{18}$.

X.

The Group $T_{8,3}$ of P_8^3 a Set of Base-Points of a Net of Quadrics.

There are in general an infinite number of projectively distinct sets of eight points in space congruent to a single set P_8^3 under a Cremona transformation which can be decomposed into a product of cubic Cremona transformations with F-points at points of P_8^3. If, however, P_8^3 is a set of base-points of a net of quadrics we can make use of the following theorem:

*If P_8^3 is a set of base-points of a net of quadrics, there are only thirty-six projectively distinct sets congruent in some order to P_8^3.**

There are then thirty-six types of congruence if no account is taken of the order of the points. If we require that P_8^3 be an ordered set we have, since P_8^3 can be ordered in $8!$ ways, $8!36$ types of congruence. The aggregate of operations transforming P_8^3 into the $8!36$ congruent sets constitute a group which we will call $T_{8,3}$. Any one of these operations is, as presupposed, the product of cubic transformations which can be obtained from a single one by a permutation of the points of P_8^3. Hence $T_{8,3}$ is generated by a cubic transformation and the symmetric group of permutations of the points of P_8^3 of order $8!$. Abstractly then $T_{8,3}$ has as generators precisely the set to which the generators of $T_{7,2}$ were shown to be equivalent in IV. $T_{8,3}$ and $T_{7,2}$ are therefore abstractly the same group.

* Coble, "Point Sets and Allied Cremona Groups," Part II, *Trans. Am. Math. Soc.*, Vol. XVII, p. 377 (45).

XI.

P_8^3 on a Cuspidal Quartic.

A cuspidal quartic curve D in space is determined by the parametric equations

$$x_1=t^4,\quad x_2=t^2,\quad x_3=t,\quad x_4=1.$$

The condition that two points t_i and t_j coincide is $t_i-t_j=0$.

The condition that four points be on a plane is $\overset{4}{\Sigma}t_i=0$.

The condition that eight points be on a quadric is $\overset{8}{\Sigma}t_i=0$.

XII.

A Net of Quadrics on P_8^3. Generators of $T_{8,8}$.

Since we wish to consider P_8^3 as the set of base-points of a net of quadrics, we determine P_8^3 by a choice of eight values t_i subject to the single condition $\overset{8}{\Sigma}t_i=0$, for the quartic above is the intersection of the quadrics

$$Q_2\equiv x_1x_4-x_2^2=0 \quad\text{and}\quad Q_3\equiv x_2x_4-x_3^2=0.$$

A third quadric on P_8^3 is

$$Q_1\equiv x_1^2+b_2x_1x_2-b_3x_1x_3+b_4x_2^2-b_5x_2x_3+b_6x_3^2-b_7x_3x_4+b_8x_4^2=0,$$

where b_i is the symmetric function of the t's of degree i. Hence:

P_8^3 determined by the choice of eight t's subject to the single condition $b_1=0$ is the set of base-points of the net of quadrics

$$y_1Q_1+y_2Q_2+y_3Q_3=0.$$

Generators of the group $T_{8,8}$ of P_8^3 determined in this way consist of the symmetric group of permutations of the eight t's together with a transformation on the t's corresponding to a cubic Cremona transformation A_{1234} with F-points at points of P_8^3, say at t_1, t_2, t_3, t_4; for the effect of A_{1234} is to send the net of quadrics into a net of quadrics, and the cuspidal quartic D into another cuspidal quartic D' whose points are named by the same parameter t. The quartic D' can be sent back into D by means of a collineation carrying the point t of D' into the point t' of D. Thus, can the transformation A_{1234} be regarded as a transformation upon the parameters t to new parameters t'.

The transformation

$$t_i'=t_i-\tfrac{1}{2}(t_1+t_2+t_3+t_4)\qquad(i=1,2,3,4),$$
$$t_j'=t_j+\tfrac{1}{2}(t_1+t_2+t_3+t_4)\qquad(j=5,6,7,8),$$

is seen to have the effect of permuting the conditions that two points coincide, that four points be on a plane as does the transformation A_{1234}, and thus gives the effect of A_{1234} upon P_8^3 in terms of the parameters t_i.

XIII.

The Quartic D^4.

If we consider y_1, y_2, y_3 as the coordinates of a point in a plane, we have by means of the net of quadrics

$$y_1Q_1+y_2Q_2+y_3Q_3=0,$$

a correspondence between the points y of a plane and the quadrics (yQ) of the net. To a pencil of quadrics or the elliptic quartic curve carrying the pencil corresponds a line of the plane. Corresponding to the quadrics of the net with a double point we will have a certain locus of points in the plane. Since in each pencil of the net are four quadrics with a double point, the locus is a quartic curve. Its equation in variables y_i found by writing the discriminant of the net is

$$D^4\equiv\begin{vmatrix} 2y_1 & b_2y_1 & -b_3y_1 & y_2 \\ b_2y_1 & 2(b_4y_1-y_2) & -b_5y_1 & y_3 \\ -b_3y_1 & -b_5y_1 & 2(b_6y_1-y_3) & -b_7y_1 \\ y_2 & y_3 & -b_7y_1 & 2b_8y_1 \end{vmatrix}$$

$$\begin{aligned}\equiv&(-4b_2^2b_6b_8+b_2^2b_7^2+4b_2b_3b_5b_8-4b_3^2b_4b_8+16b_4b_6b_8-4b_4b_7^2)y_1^4\\ &+(-2b_2b_5b_7+4b_3^2b_8+4b_3b_4b_7-16b_6b_8+4b_7^2)y_1^3y_2\\ &+(4b_2^2b_8-2b_2b_3b_7-16b_4b_8+4b_5b_7)y_1^3y_3+(-4b_3b_7-4b_4b_6+b_5^2)y_1^2y_2^2\\ &+(4b_2b_6-2b_3b_5+16b_8)y_1^2y_2y_3+(b_3^2-4b_6)y_1^2y_3^2+4b_6y_1y_2^3\\ &+4b_4y_1y_2^2y_3-4b_2y_1y_2y_3^2+4y_1y_3^3-4y_2^3y_3=0.\end{aligned}$$

XIV.

Complete System for $T_{8,3}$.

The quartic D^4 consists of precisely the same terms as does C^4 above. The flex tangent now is the map of the quartic D which is unaltered by the operations of $T_{8,3}$. If we choose as base quadrics of the net quadrics covariant with D, since the discriminant is an invariant of the net, its coefficients will be invariants of $T_{8,3}$. Since the quartic D maps into the flex tangent, we need not determine these quadrics, but have only to choose in the plane a triangle of reference covariant with the flex. The lines are chosen in the same way as for C^4. The transformation is therefore of the same type as (5), removes the same terms from D^4 as (5) does from C^4 and is

$$y_1=3y_1',\quad y_2=b_4y_1'+3y_2',\quad y_3=3b_6y_1'+3y_3'.$$

The transformed expression for D^4 is

$$3B_{18}y_1'^4+27B_{14}y_1'^3y_2'+3B_{12}y_1'^3y_3'+81B_{10}y_1'^2y_2'^2+27B_8y_1'^2y_2'y_3'$$
$$+27B_6y_1'^2y_3'^2+81B_2y_1'y_2'y_3'^2+324y_1'y_3'^3-324y_2'^3y_3'=0,$$

where B_i is an invariant of $T_{8,3}$ of degree i, whose explicit expressions are as follows:

$$\begin{aligned}
B_2 &=-4b_2,\\
B_6 &=-4b_2b_4+3b_3^2+24b_6,\\
B_8 &=-12b_2b_6-6b_3b_5+4b_4^2+48b_8,\\
B_{10} &=-4b_3b_7+b_5^2,\\
B_{12} &= 54b_3^2b_6-18b_3b_4b_5+8b_4^3+108b_6^2,\\
B_{14} &=-6b_2b_5b_7-6b_3b_5b_6+12b_3^2b_8+4b_3b_4b_7+2b_4b_5^2+12b_7^2,\\
B_{18} &= 27b_2^2b_7^2+108b_2b_3b_5b_8-54b_2b_3b_6b_7-18b_2b_4b_5b_7-72b_3^2b_4b_8+27b_3^2b_6^2\\
&\quad +24b_3b_4^2b_7-18b_3b_4b_5b_6+3b_4^2b_5^2-72b_4b_7^2-108b_5^2b_8+108b_5b_6b_7.
\end{aligned}$$

It is to be noted that great simplicity is gained in the complete system when the group is represented in eight variables whose sum is zero. This is to be expected since every term in which b_1 enters vanishes. Moreover, the notation for the double tangents is symmetrical.

The lines of D^4 arise from the pencils of quadrics of the net. The quadrics in a pencil with a double point correspond to the meets of the line with D^4. To a pencil of quadrics such that the double points have coincided in pairs corresponds a double tangent of D^4. Such a pencil of the net can be found by requiring that it contain the line t_it_j. To do this we have only to require that the quadric contain a point of the line t_it_j other than t_i and t_j. That is, the linear condition on the y''s is

$$(3\sigma_6+3s_2\sigma_4+3s_1s_2\sigma_3+3s_1^2s_2\sigma_2-s_1^2b_4-3b_6)\,y_1'-3s_1^2y_2'-3y_3'=0,$$

where s_k is the symmetric function of t_i and t_j of degree k, and σ_k the symmetric function of the remaining six t's. This is the condition in the plane that the point y be on a double tangent, that is, it is the equation of the double tangent corresponding to a choice of two of the eight points of P_8^3. The twenty-eight double tangents are thus all accounted for and are all of one type (ij).

Proof of Pohlke's Theorem and its Generalizations by Affinity.

By Arnold Emch.

1. *Introduction.*

The purpose of this paper is to show how Pohlke's Theorem, its generalizations, and some related propositions may be proved in a comprehensive manner by making use of affine collineations in space.

The theorem was first published without proof in the first part of Pohlke's "Descriptive Geometry" in 1860, and may be stated as follows:

Three straight line segments of arbitrary length in a plane, drawn from a point and making arbitrary angles with each other, form a parallel projection of three equal segments drawn from the origin on three rectangular coordinate-axes; however, only one of the segments, or one of the angles can vanish.

The first elementary rigorous proof of the fundamental theorem of axonometry, as Pohlke's Theorem is sometimes called, was given by H. A. Schwarz.* Subsequently numerous other proofs of the theorem and, in a few instances, of its generalization for an oblique system of coordinate-axes were given.†

---

* *Crelle's Journal*, Vol. LXIII (1864), pp. 309–314, "Elementarer Beweis des Pohlkeschen Fundamentalsatzes der Axonometrie.

† von Deschwanden, who received his knowledge of the theorem from Steiner on one of the latter's visits to Zürich, gave a proof in the *Vierteljahrsschrift of the Naturforschende Gesellschaft in Zürich*, Vol. VI (1861), pp. 254–284, which, however, was not entirely satisfactory.

In the same volume, pp. 358–367, Kinkelin gave an analytic proof.

In Vol. XI, pp. 350–358, of the same publication, Reye, by means of projective geometry, generalized the theorem for oblique coordinates.

Among others who gave purely geometric demonstrations of the theorem, and constructive solutions of the problem involved may be mentioned:

Pelz, *Wiener Berichts*, Vol. LXXVI, II (1877), pp. 123–128.

Peschka, *Ibid.*, Vol. LXXVIII, II (1879), pp. 1043–1055.

Mandel, *Ibid.*, Vol. XCIV, II (1886), pp. 60–65.

Ruth, *Ibid.*, Vol. C, II (1891), pp. 1088–1092.

Schur, *Mathematische Annalen*, Vol. XXV (1885), pp. 596–597.

Schur, *Crelle's Journal*, Vol. CXVII (1896), pp. 474–475.

Küpper, *Mathematische Annalen*, Vol. XXXIII (1889), pp. 474–475.

Beck, *Crelle's Journal*, Vol. CVI (1890), pp. 121–124.

Schilling, *Zeitschrift für Mathematik und Physik*, Vol. XLVIII (1903), pp. 487–494.

Loria, *Vorlesungen über darstellende Geometrie*, Vol. I (1907), pp. 190–194,

Grossmann, *Darstellende Geometrie* (1915), pp. 26–29,

and various other well-known texts on descriptive geometry make use of the theorem in the discussion of axonometry.

I shall first investigate some of the properties of affine collineations in space, as far as they are related to the problem involved. Based upon these properties it will then not be difficult to prove Pohlke's and a number of similar theorems.

2. *Definition and General Properties of Affinity.*

Let OX, OY, OZ and $O'X'$, $O'Y'$, $O'Z'$ be two systems of coordinates, which, for the sake of definiteness, we assume as orthogonal; then the two spaces are defined as related by affinity when their coordinates are connected by the substitution

$$S \equiv \left\{ \begin{array}{l} x' = a_0 + a_1 x + a_2 y + a_3 z, \\ y' = b_0 + b_1 x + b_2 y + b_3 z, \\ z' = c_0 + c_1 x + c_2 y + c_3 z. \end{array} \right\} \tag{1}$$

The classification * of affinities depends upon the properties of the matrix

$$\left\| \begin{array}{llll} a_1 - 1 & a_2 & a_3 & a_0 \\ b_1 & b_2 - 1 & b_3 & b_0 \\ c_1 & c_2 & c_3 - 1 & c_0 \end{array} \right\|. \tag{2}$$

They form a projective twelve-parameter group and leave the plane at infinity invariant. Parallel planes and parallel lines are transformed into parallel planes and parallel lines. Of particular importance for our purpose is the case where the rank of matrix (2) is 1, so that the values of all its determinants of orders 3 and 2 vanish. The geometric meaning of this case is that the points of a certain plane s are left invariant, and that corresponding points lie on lines all parallel to a definite direction. Moreover, when P and P' are corresponding points and P_1 is the intersection of PP' with s, then $P'P_1 : PP_1 =$ constant. By a translation we can always make a_0, b_0, c_0 vanish, so that the origin $O \equiv O'$ becomes an invariant point. In this case the special affinity H, whose matrix is of rank 1, may always be written in the form

$$H \equiv \left\{ \begin{array}{l} x' = x + \lambda_1 (x + py + qz), \\ y' = y + \lambda_2 (x + py + qz), \\ z' = z + \lambda_3 (x + py + qz), \end{array} \right\} \tag{3}$$

where $x + py + qz = 0$ is the plane, all of whose points are invariant. Corresponding points lie on parallel lines whose direction is determined by the constant ratios:

$$(x' - x)/(z' - z) = \lambda_1/\lambda_3, \quad (y' - y)/(z' - z) = \lambda_2/\lambda_3.$$

* *Pascal's Repertorium*, Vol. II (2d ed.), pp. 100–101.

The line joining any two distinct corresponding points $P'(x', y', z')$, $P(x, y, z)$ cuts s in a point P_1 so that

$$P'P_1/PP_1 = 1+\lambda_1+\lambda_2 p+\lambda_3 q = \text{const.},$$

as stated above. This constant is also equal to the value of the determinant

$$\Delta = \begin{vmatrix} 1+\lambda_1 & \lambda_1 p & \lambda_1 q \\ \lambda_2 & 1+\lambda_2 p & \lambda_2 q \\ \lambda_3 & \lambda_3 p & 1+\lambda_3 q \end{vmatrix}$$

of the substitution H. When $\Delta=0$, then the homologous affinity H becomes a parallel projection on the plane s. The affinity is, in this case, singular.

Through every point $P(x, y, z)$ of an affinity S there is just one system of three mutually orthogonal planes which is transformed into such an orthogonal system through $P'(x', y', z')$. If these two systems are chosen as coordinate planes, S assumes the simple form

$$D \equiv x' = ax, \quad y' = by, \quad z' = cz, \tag{4}$$

which is called a dilatation. As the two coordinate systems (which we may assume as having both the same sense) may be brought to coincidence by a rotation R we have the well-known

THEOREM I. *Every affinity S may be considered as the product of a rotation R and a dilatation D, so that $S=RD$.**

3. *Homologous Affinity.*

We shall now consider in particular the affinity of type H. Such an affinity is also determined by two tetrahedrons whose corresponding points lie on four non-coplanar parallel lines. The planes of corresponding faces, and of corresponding planes, in general, meet in lines of a fixed plane s, the plane of homology. The parallel lines joining corresponding points pass through the same infinite point, called center of homology. We shall call H an homologous affinity.

The question is, whether it is possible to represent S in the form $S=R_1D_1H$, in which the substitution

$$D_1 \equiv x' = \rho x, \quad y' = \rho y, \quad z' = \rho z, \tag{5}$$

is a similitude.

For this purpose we first prove

THEOREM II. *There always exist homologous affinities by which any ellipsoid is transformed into a sphere, and conversely.*

Let $$x^2/a^2+y^2/b^2+z^2/c^2=1 \tag{6}$$

* *Pascal's Repertorium, loc. cit.* Koenigs, "Leçon de cinématique" (1897), pp. 394–405.

be any ellipsoid, and assume $a > b > c$. The two systems of circular sections are parallel to the diametral planes

$$z = \pm \frac{c}{a} \sqrt{\frac{a^2 - b^2}{b^2 - c^2}} \cdot x. \tag{7}$$

Consider the plane s, whose equation is obtained from (7) by choosing the — sign, and which cuts the ellipsoid in a circle. Through this as a great circle pass a sphere, whose equation will be

$$x^2 + y^2 + z^2 = b^2. \tag{8}$$

It is easily shown that the ellipsoid (6) and the sphere (8) are inscribed in two right circular cylinders whose axes are in the xz-plane and have the slopes

$$m = \pm \sqrt{(b^2 - c^2)/(a^2 - b^2)}. \tag{9}$$

Denoting the coordinates of a point of the ellipsoid by x', y', z', and of a point on the sphere by x, y, z, and considering the cylinder obtained by taking the + sign in (9), it is found that the ellipsoid results from the sphere by the homologous affinity H, defined by

$$H^{-1} \equiv \left\{ \begin{array}{l} x = x' - \dfrac{(ac + b^2)\sqrt{(a^2 - b^2)(b^2 - c^2)}}{ac\{a(b^2 - c^2) + c(a^2 - b^2)\}} \cdot \left\{ c\sqrt{\dfrac{a^2 - b^2}{b^2 - c^2}} \cdot x' + az' \right\}, \\ y = y', \\ z = z' - \dfrac{(ac + b^2)(b^2 - c^2)}{ac\{a(b^2 - c^2) + c(a^2 - b^2)\}} \cdot \left\{ c\sqrt{\dfrac{a^2 - b^2}{b^2 - c^2}} \cdot x' + az' \right\}, \end{array} \right\} \tag{10}$$

with s as the invariant plane, and the direction in the xz-plane with the slope $m = +\sqrt{(b^2 - c^2)/(a^2 - b^2)}$ as that of the infinite center of homology. Two corresponding points P' and P of H are joined by a line cutting s in P_1, so that $P'P_1/PP_1 = -\dfrac{ac}{b^2} =$ constant. The slope $m = -\sqrt{(b^2 - c^2)/(a^2 - b^2)}$ in (9) determines another homologous affinity with the same property, which is symmetrical with the first, with respect to the yz-plane. Now, an affinity transforms conjugate poles and polar planes and triplets of conjugate diameters of a quadric into corresponding poles and polar planes and triplets of conjugate diameters of the transformed quadric. Consequently, by the homologous affinity H^{-1} any three conjugate diameters of the ellipsoid (6) are transformed into three conjugate diameters of the sphere (8), which, as such, are orthogonal to each other. Likewise, any three conjugate radii OA', OB', OC' of the ellipsoid are transformed into three rectangular radii OA, OB, OC of the sphere. The lines AA', BB', CC' cut the sphere in three other points $A_1B_1C_1$, so that also OA_1, OB_1, OC_1 are orthogonal. The slope $-m$ determines two other

orthogonal trihedrals on the sphere, so that their extremities lie twice in sets on three parallel lines through A', B', C'.

But any three non-coplanar lines $A'A'_{-1}$, $B'B'_{-1}$, $C'C'_{-1}$, which bisect each other at O, as conjugate diameters uniquely determine an ellipsoid with O as a center. By Chasle's * or other † well-known methods the three rectangular-conjugate diametral planes, and the orthogonal-conjugate diameters, axes, may be constructed. Using these as coordinate axes, and denoting the semi-axes in the order of their magnitude by a, b, c, the equation of the ellipsoid may be written in the form (6). Then, by the method explained above we may construct the four orthogonal trihedrals on the corresponding affine sphere.

4. *A Certain Composition of Affinity.*

It is now possible to answer the question concerning the representation of a general affinity S in the form $S=R_1D_1H$.

According to Theorem I, let R be the rotation, and D the dilatation, so that $S=RD$ carries a point (x, y, z) into the point (x', y', z'). Around these points determine the corresponding orthogonal systems of coordinate axes, so that by S the sphere K, $x^2+y^2+z^2=1$, is transformed into the ellipsoid E, $x'^2/a^2+y'^2/b^2+z'^2/c^2=1$, with $a>b>c$. According to the method explained above construct the sphere K_2, so that E is obtained from K_2 by a homologous affinity H. Determine the orthogonal coordinate system G_2 through the center of K_2, corresponding to the orthogonal system through the center of E. The equation of K_2 with respect to G_2 is $x_2^2+y_2^2+z_2^2=b^2$. To G_2 apply the similitude

$$D_1^{-1}\equiv x_1=x_2/b, \quad y_1=y_2/b, \quad z_1=z_2/b. \tag{11}$$

Finally, by a definite rotation R_1^{-1} the system $G_1(x_1, y_1, z_1)$ is transformed into the original system $G(x, y, z)$. Conversely, by R_1 the sphere $x^2+y^2+z^2=1$ is transformed into the sphere K_1, $x_1^2+y_1^2+z_1^2=1$. By D_1, K_1 is transformed into the sphere K_2, $x_2^2+y_2^2+z_2^2=b^2$. By H, K_2 is transformed into the ellipsoid $x'^2/a^2+y'^2/b^2+z'^2/c^2=1$. With this the identity $S\equiv RD\equiv R_1D_1H$ is proved for non-singular affinities.

It is also true for a singular affinity S_s, in which all points (x, y, z) are transformed into points (x', y', z') which lie in a plane. If we choose this as the plane s, the dilatation D_s will have the form

$$D_s\equiv x'=ax, \quad y'=by, \quad z'=0z. \tag{12}$$

* Beck, *loc. cit.*

† Fiedler, *Darstellende Geometrie*, Vol. II (1885), pp. 329–330.

The homologous affinity H becomes a parallel projection upon a plane s, defined by

$$H_s \equiv x' = x - \frac{\sqrt{a^2-b^2}}{b} \cdot z, \quad y'=y, \quad z'=0, \tag{13}$$

as is easily found from (10), by solving for x', y', z', and letting $\lim\,(c)=0$. The ellipsoid becomes an infinitely thin disk E_s in the plane s, whose contour has the equation $x'^2/a^2+y'^2/b^2=1$, and we find again $S_s=R_1 \cdot D_1 \cdot H_s$. Hence

THEOREM III. *Every general affinity in space is the product of a rotation, a similitude, and an homologous affinity. In case of a singular affinity, in which all points are transformed into points of a plane, the homologous affinity becomes a parallel projection upon this plane.*

5. *Pohlke's Theorem and its Generalization.*

Chasle's construction of the axes of an ellipsoid still holds when the three conjugate diameters $A'A'_{-1}$, $B'B'_{-1}$, $C'C'_{-1}$ are coplanar. As before, the three diametral planes cut the degenerate ellipsoid E_s in three ellipses $(A'B')$, $(B'C')$, $(C'A')$, which are inscribed in three parallelograms as shown in Fig. 1.

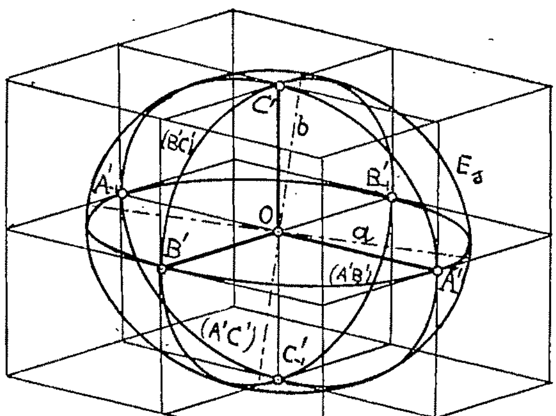


FIG. 1.

These ellipses are inscribed in an ellipse E_s in s, whose half-axes we denote by a and b $(a>b)$, and whose lines we choose as x'- and y'-axes, and the line through the center of E_s, perpendicular to s, as the z'-axis. Then the equation of E_s is precisely that given above as $x'^2/a^2+y'^2/b^2=1$. The sphere K with the radius b, concentric with E_s, is now projected into E_s by a parallel projection H_s as defined by (13). Conversely, by the same formulas for H_s, and geometrically, it is easily verified, that to the ellipses $(A'B')$, $(B'C')$, $(C'A')$,

and their circumscribed parallelograms, correspond on K three great circles, whose planes are mutually perpendicular, and their circumscribed squares. To the complete rhombohedral lattice-work, circumscribed and inscribed to the ellipsoid as before, corresponds a cubical lattice-work connected in the same manner to the sphere, so that the rhombohedral is the parallel-projection (H_s)

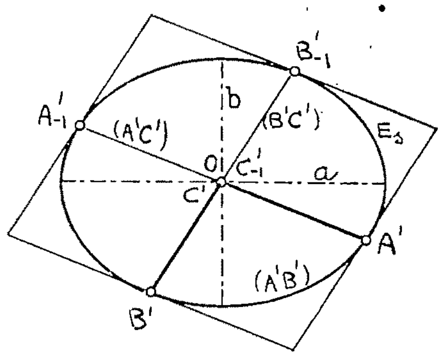


FIG. 2.

of the cubical lattice work. In this manner to the three distinct conjugate coplanar radii OA', OB', OC', correspond the three equal orthogonal radii OA, OB, OC. There are, in general, again four such sets of orthogonal radii.

The proposition is still true when one of the three coplanar radii, say OC', vanishes. The ellipse $(A'B')$ has $A'A'_{-1}$ and $B'B'_{-1}$ as conjugate diameters, while the ellipses $(B'C')$ and $(C'A')$ coincide with the segments $B'B'_{-1}$ and $A'A'_{-1}$, Fig. 2. The contour ellipse E_s coincides with the ellipse $(A'B')$. The

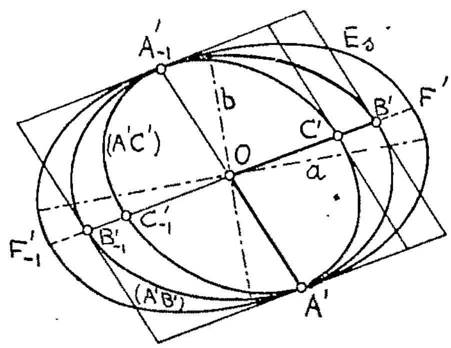


FIG. 3.

homologous sphere may be constructed precisely as before, so that to the ellipses $(A'B')$, $(B'C' = \text{degenerate})$, $(C'A' = \text{degenerate})$ correspond again three orthogonal great circles on the sphere K, and to the coplanar conjugate semi-diameters OA', OB', $OC' = 0$, three orthogonal radiï of K.

When one of the angles, say the one between $B'B'_{-1}$ and $C'C'_{-1}$ is zero, Fig. 3, then the ellipses $(A'B')$, $(A'C')$ may be constructed as in the general

case. The ellipse $(B'C')$ degenerates into a straight line segment $F'F'_{-1}$, so that $|OF'| = |OF'_{-1}| = \sqrt{\overline{OB'}^2 + \overline{OC'}^2}$. In this case E_s is the ellipse which has $A'A'_{-1}$ and $F'F'_{-1}$ as conjugate diameters, and which with the ellipses $(A'B')$, $(B'C')$, $(C'A')$, and the corresponding circumscribed parallelograms and conjugate diameters may be considered as the parallel projection H_s of a sphere K with three orthogonal great circles and the attached cubic lattice-work. In every case there exist equiaxial-orthogonal trihedrals ($OA = OB = OC$; $\sphericalangle AOB = \sphericalangle BOC = \sphericalangle COA = 90°$) of which OA', OB', OC', whether coplanar or not, form a parallel projection. Hence, we may state Pohlke's Theorem in a generalized form.

THEOREM IV. *The vertex and the extremities of any three concurrent, coplanar or non-coplanar, straight line segments in space always lie in a definite order on four parallel lines through the vertex and the extremities of an equiaxial-orthogonal trihedral. In general, there are two distinct sets of four parallel lines each, and four sets of orthogonal trihedrals with this property. Not more than one segment, and not more than one angle between the segments of the given trihedral may vanish.*

As a general affinity S depends on twelve independent parameters, it is always possible to determine uniquely an affinity S in which any two proper tetrahedrons T' and T correspond to each other in a definite order. For example, $P'_1P'_2P'_3P'_4$ to $P_1P_2P_3P_4$. But we have proved that $S = R_1D_1H$, so that T' results from T by a rotation, followed by a similitude, and finally by an homologous affinity. When $P'_1P'_2P'_3P'_4$ are coplanar, then the substitution H becomes a parallel projection H_s, and S is a singular affinity, for which the determinant of the substitution vanishes. The result may be stated as

THEOREM V.* *If any two proper tetrahedrons $P'_1P'_2P'_3P'_4$ and $P_1P_2P_3P_4$ are given, it is always possible to determine a tetrahedron $P''_1P''_2P''_3P''_4$ similar (eventually congruent) to $P_1P_2P_3P_4$, so that the lines joining P'_1 and P''_1, P'_2 and P''_2, P'_3 and P''_3, P'_4 and P''_4 are parallel. This is still true when the points $P'_1P'_2P'_3P'_4$ form a proper plain quadrangle, or also when the segments $P'_4P'_1$, $P'_4P'_2$, $P'_4P'_3$ and the angles formed by them are subject to the necessary and sufficient conditions of Pohlke's Theorem.*

This theorem clearly contains Pohlke's and Reye's theorems as special cases.

---

*The first part of this theorem concerning two proper tetrahedrons has also been proved by Hurwitz in a recent communication to the Swiss Mathematical Society, an abstract of which in *L'Enseignement Mathématique* reached the author several months after this paper was sent to the AMERICAN JOURNAL OF MATHEMATICS.

6. *Related Theorems.*

From the connection between the rhombohedral and cubical lattice-works discussed above, we deduce without difficulty

THEOREM VI. *A plain hexagon with three pairs of parallel, opposite sides, with the sides of each pair equal, may always be considered as the contour of a parallel projection of a cube. The net of six parallelograms constructed with each two adjacent sides of the hexagon as a pair of adjacent sides of a parallelogram, is the projection of the edges of the cube.*

Completing the rhombohedral lattice-works, determined by P_4P_1, P_4P_2, P_4P_3 and $P_4'P_1'$, $P_4'P_2'$, $P_4'P_3'$ as clinographic semi-axes of the rhombohedrons, and inscribing ellipsoids into these, with the clinographic axes in each case as triplets of conjugate diameters, we find

THEOREM VII. *If any two parallelopipeds (rhombohedrons) π' and π are given, it is always possible to find a parallelopiped π'' similar (eventually congruent) to π, so that corresponding vertices of π' and π'' lie on eight parallel lines (eventually counting multiplicities properly).*

In a similar manner we have

THEOREM VIII. *If any two ellipsoids E' and E are given, it is always possible to find an ellipsoid E'' similar (eventually congruent) to E, so that E' and E'' are inscribed to the same elliptic (circular) cylinder.*

Likewise as a special case of the foregoing,

THEOREM IX. *The contour of a parallel projection of any given ellipsoid upon a plane may be similar to any given ellipse.*

Finally,

THEOREM X. *It is always possible to circumscribe two (may be coincident) right circular cylinders to any ellipsoid.*

Arithmetical Theory of Certain Hurwitzian Continued Fractions.

By D. N. Lehmer.

Introduction.

The following investigation is the outcome of the discovery, made some three years ago, of the curious fact that the denominator of the convergent of order $3n$ in the regular continued fraction which represents the base of Naperian logarithms is divisible by n. It was later found that the same is true of the denominators of the convergents of order $3n-2$ and $3n-6$, and of the numerator of the convergent of order $3n-3$. Further, the convergents were found to recur with a period of $3n$ terms, or of $6n$ terms according as n is even or odd.

These theorems, discovered empirically, turned out to be remarkably intractable, and, although, a year ago, a method was discovered of establishing them, it would not apply to other continued fractions of the same general type for which the same or similar theorems seemed to hold.

The discovery of the regular continued fraction

$$e=(2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, \ldots.),$$

for the base of Naperian logarithms is credited to Roger Cotes,* but to Euler † is due the rediscovery of it, and the general proof of the law of the successive partial quotients by means of the solution of Riccati's equation. Euler also found other remarkable continued fractions such as

$$e=(1, 1, 1, 1, 5, 1, 1, 9, 1, 1, 14, 1, 1, \ldots.),$$

$$(e^{\frac{1}{s}}+1)/(e^{\frac{1}{s}}-1)=(2s, 6s, 10s, 14s, \ldots.),$$

$$e^{\frac{1}{s}}=(1, s-1, 1, 1, 3s-1, 1, 1, 5s-1, 1, 1, \ldots.).$$

Hurwitz ‡ has studied a very general type of continued fraction, to which the above fractions all belong. He makes use of the notation

$$(q_1, q_2, \ldots, q_r, \overline{f_1(m), f_2(m), f_3(m), \ldots, f_k(m)}), \quad (m=0, 1, 2, 3, \ldots.),$$

for the continued fraction whose partial quotients are

$$q_1, q_2, \ldots, q_r, f_1(0), f_2(0), \ldots, f_k(0), f_1(1), f_2(1), \ldots, f_k(1), f_1(2), f_2(2), \ldots,$$

* Cotes, "Logometria," *Phil. Trans.*, London (1714), Vol. XXIX, p. 5.

† Euler, *Comm. Acad. Petrop.* (1737), p. 121, edition of 1744.

‡ Hurwitz, *Vierteljahrschrift der Naturforschenden Gesellschaft in Zurich* (1896), Vol. XLI, p. 34.

where the q's are rational, and with the possible exception of q_1, all positive. The functions f are rational, integral functions whose degrees may some or all be zero. If, however, the degrees are all zero, the fraction becomes an ordinary periodic continued fraction. If the highest degree found among any of the functions is m, the fraction is said to be of the m-th order. An ordinary periodic continued fraction is thus a Hurwitzian fraction of order zero. Written in this notation the above fractions of Euler read:

$$e=(2, \overline{1, 2m+2, 1}), \quad (m=0, 1, 2, 3, \ldots\ldots),$$
$$(e^{\frac{1}{s}}+1)/(e^{\frac{1}{s}}-1)=[\overline{(4m+2)s}], \quad (m=0, 1, 2, 3, \ldots\ldots),$$
$$e^{\frac{1}{s}}=[\overline{1, (2m+1)s, 1}], \quad (m=0, 1, 2, 3, \ldots\ldots),$$

Hurwitz has shown that if the irrational numbers ξ and η are connected by the equation $\xi=(\alpha\eta+\beta)/(\gamma\eta+\delta)$, where $\alpha, \beta, \gamma, \delta$ are integers such that $\alpha\delta-\beta\gamma$ is not zero, then if the regular continued fraction for η is of the Hurwitzian type, so also is that for ξ, and the functions f appearing in each expansion are of the same degrees, with the possible exception of those of zero degree which may appear in one and not in the other.

In this paper we shall deal with Hurwitzian fractions in which the functions f are all of degree zero except one, and that one is of the first degree. We shall write the fraction in the form:

$$(q_1, q_2, \ldots\ldots, q_r, \overline{a_1, a_2, a_3, \ldots\ldots, a_{k-1}, bm+c}), \quad (m=1, 2, 3, \ldots\ldots),$$

and taking first the case where the q's are all absent, we show that, with certain interesting exceptions, it is true for all such fractions that the numerator of the convergent of order $2nk-1$, and the denominator of the convergent of order $2nk$ are divisible by n, while the numerator of the convergent of order $2nk$, and the denominator of the convergent of order $2nk-1$ are congruent modulo n to $(-1)^{nk-1}$, so that the series of convergents repeat themselves, modulo n after $4nk$ terms, or after $2nk$ terms according as nk is even or odd.

Exceptions to this rule occur when b is congruent to zero, modulo n, or when the numerator or denominator of the convergent of order $k-1$ is congruent to zero, modulo n. The period of the convergents is not so simply stated for these cases.

The laws for the period of the fraction when the q's are actually present are easily obtainable.

In this paper we are not considering questions of convergence or divergence of continued fractions. Certain of the fractions involved are closely related to those called "semiregular" whose convergence has been studied by

Tietze,* and the rules for determining convergence or divergence of semi-regular continued fractions may be modified to apply to them. We are concerned here with the successive values of the numerators and denominators of the convergents, and not with the existence or non-existence of a limiting value to those convergents. The theorems obtained have to do with numbers which satisfy certain difference equations of the second order, and are thus, as Professor Birkhoff has remarked, extensions of Wilson's and Fermat's theorems, which have to do with numbers which satisfy the difference equations $u_{n+1}=nu_n$ and $u_{n+1}=au_n$ respectively.

I.

Let A_m/B_m be the m-th convergent of the continued fraction

$$\overline{(a_1, a_2, a_3, \ldots, a_{k-1}, \mu b)}, \qquad (\mu=1, 2, 3, \ldots), \tag{1}$$

where all the partial quotients are positive or negative integers or zero. Let also A'_m/B'_m be the m-th convergent of the fraction

$$\overline{(a_{k-1}, a_{k-2}, \ldots, a_2, a_1, -b\mu-2M)}, \qquad (\mu=1, 2, \ldots), \tag{2}$$

where
$$M=(A_{k-2}+B_{k-1})/A_{k-1}, \tag{3}$$

and we will suppose that A_{k-1} is not zero. We will show by complete induction that the following equations hold:

$$A_{rk-1}=A'_{rk-1}(-1)^{r-1}, \tag{4}$$

$$A_{rk}=(A'_{rk}+MA'_{rk-1})(-1)^r. \tag{5}$$

By using equation (4) we may interchange A and A' in (5).

To start the proof it is necessary to show that the formulae hold for $r=1$. The fractions $(a_1, a_2, a_3, \ldots, a_{k-1})$ and $(a_{k-1}, \ldots, a_2, a_1)$ are inverse. (See Perron, "Die Lehre von den Kettenbrüchen," p. 32). We have, therefore, $A'_{k-1}/A'_{k-2}=(a_1, a_2, \ldots, a_{k-1})=A_{k-1}/B_{k-1}$, and since the fractions are in their lowest terms,

$$A_{k-1}=A'_{k-1}, \tag{6}$$

$$B_{k-1}=A'_{k-2}; \tag{7}$$

and similarly,

$$A_{k-2}=B'_{k-1}, \tag{8}$$

$$B_{k-2}=B'_{k-2}. \tag{9}$$

Equation (4) therefore holds when $r=1$, by (6). Also, from the definition of M we have, using (6) again, $MA'_{k-1}=A_{k-2}+B_{k-1}$, so that when $r=1$, (5) becomes

$$A_k=-(A'_k+A_{k-2}+B_{k-1}). \tag{10}$$

---

* Tietze, *Math. Ann.* (1911), Vol. LXX.

But from the definitions of the continued fractions themselves we have

$$A_k = bA_{k-1} + A_{k-2}, \tag{11}$$

and

$$A'_k = -(b+2M)A'_{k-1} + A'_{k-2}; \tag{12}$$

or, using (6) and (7),

$$A'_k = -(bA_{k-1} + 2A_{k-2} + B_{k-1}). \tag{13}$$

Substituting (11) and (13) in (10) it is found that equation (5) is true when $r=1$.

We now show that if equations (4) and (5) are assumed to hold for all values of r up to and including $r=n$, they must also hold for $r=n+1$.

By the fundamental formula of continued fractions (see Perron, *loc. cit.*, p. 14),

$$A_{(n+1)k-1} = A_{k-1}A_{nk} + B_{k-1}A_{nk-1}, \tag{14}$$

and assuming formulae (4) and (5) for $r=n$ this gives

$$A_{(n+1)k-1} = (-1)^n A_{k-1}(A'_{nk} + MA'_{nk-1}) + (-1)^{n-1}A'_{nk-1}B_{k-1},$$

and, recalling the definition of M, this becomes

$$A_{(n+1)k-1} = (-1)^n (A_{k-1}A'_{nk} + A_{k-2}A'_{nk-1}),$$

and using (6) and (8) this is $A_{(n+1)k-1} = (-1)^n(A'_{k-1}A'_{nk} + B'_{k-1}A'_{nk-1})$, which, by the fundamental formula (14), gives

$$A_{(n+1)k-1} = (-1)^n A'_{(n+1)k-1}, \tag{15}$$

which is formula (4) for $r=n+1$.

Suppose now that formula (5) holds for $r=n$, so that

$$A_{nk} = (-1)^n(A'_{nk} + MA'_{nk-1}).$$

Multiply both sides of this equation by A_{k-2}, and add $B_{k-2}A_{nk-1}$ to the left-hand side, and the equal expression, $(-1)^{n-1}B_{k-2}A'_{nk-1}$ to the right. We thus obtain

$$A_{k-2}A_{nk} + B_{k-2}A_{nk-1} = (-1)^n(A_{k-2}A'_{nk} + MA_{k-2}A'_{nk-1} - B_{k-2}A'_{nk-1}).$$

Using (6), (7), (8) and (9), we can throw this into the form

$$A_{k-2}A_{nk} + B_{k-2}A_{nk-1} = (-1)^n[M(A'_{k-1}A'_{nk} + B'_{k-1}A'_{nk-1}) - (A'_{k-2}A'_{nk} + B'_{k-2}A'_{nk-1})].$$

But by the fundamental formulae of continued fractions (Perron, *loc. cit.*, p. 14), this last equation may be written $A_{(n+1)k-2} = (-1)^n(MA_{(n+1)k-1} - A_{(n+1)k-2})$. To the left side of this equation add the term $(n+1)bA_{(n+1)k-1}$, and to the right side add the term $(-1)^n(n+1)bA'_{(n+1)k-1}$, which by equation (15) is legitimate, and the result is

$$(n+1)bA_{(n+1)k-1} + A_{(n+1)k-2}$$
$$= (-1)^{n+1}[-(n+1)bA'_{(n+1)k-1} - MA'_{(n+1)k-1} + A'_{(n+1)k-2}]. \tag{16}$$

But from the recurrent relation connecting the numerators in the continued fraction we have $A_{(n+1)k}=(n+1)A_{(n+1)k-2}+A_{(n+1)k-2}$, and

$$A'_{(n+1)k}=-[(n+1)b+2M]A'_{(n+1)k-1}+A'_{(n+1)k-2}.$$

Putting these values in (16) we get $A_{(n+1)k}=(-1)^{n+1}(A'_{(n+1)k}+MA'_{(n+1)k-1})$, which is equation (5) when $r=n+1$. If then (4) and (5) are true for $r=n$, they must be true for $r=n+1$. But they have been shown to hold for $r=1$, therefore they hold in general.

II.

Consider now the continued fraction

$$(x,\, a_{k-1},\, a_{k-2},\, \ldots,\, a_2,\, a_1,\, y,\, \overline{a_{k-1},\, a_{k-2},\, \ldots,\, a_2,\, a_1,\, -b\mu-2M}), \qquad (\mu=1,2,\ldots), \tag{17}$$

where
$$x=(B_{k-1}-A_{k-2})/A_{k-1}, \tag{18}$$

and
$$y=-(2B^2_{k-1}+A_{k-1}B_k+A_{k-1}B_{k-2})/A_{k-1}B_{k-1}. \tag{19}$$

We assume that A_{k-1} and B_{k-1} are different from zero. We denote the $(m+1)$st convergent of this fraction by A''_m/B''_m, and show by complete induction that the following equations hold:

$$B_{rk-1}=A''_{rk-1}(-1)^{r-1}, \tag{20}$$

$$B_{rk}=(A''_{rk}+MA''_{rk-1})(-1)^r. \tag{21}$$

Equation (21) may be written, using (20),

$$A''_{rk}=(B_{rk}+MB_{rk-1})(-1)^r. \tag{22}$$

We first show that the formulae hold when $r=1$. We have $A''_{k-1}/B''_{k-1}-x=(0,\, a_{k-1},\, a_{k-2},\, \ldots,\, a_1)$, or $B''_{k-1}/(A''_{k-1}-xB''_{k-1})=(a_{k-1},\, \ldots,\, a_1)=A'_{k-1}/B'_{k-1}=A_{k-1}/A_{k-2}$, and since all fractions are in their lowest terms,

$$B''_{k-1}=A_{k-1}, \tag{23}$$

$$A''_{k-1}-xB''_{k-1}=A_{k-2}. \tag{24}$$

From these two equations with (18) we get at once

$$A''_{k-1}=B_{k-1}, \tag{25}$$

which is formula (20) when $r=1$.

In the same way, using the relations $B''_{k-2}/(A''_{k-2}-xB''_{k-2})=(a_{k-1},\, \ldots,\, a_2)=A'_{k-2}/B'_{k-2}=B_{k-1}/B_{k-2}$, we get

$$B''_{k-1}=B_{k-1}, \tag{26}$$

$$A''_{k-2}=xB_{k-1}+B_{k-2}, \tag{27}$$

whence, using the defining equation for x again, and remembering from the general theory of continued fractions that

$$A_{k-1}B_{k-2}-B_{k-1}A_{k-2}=(-1)^{k-1}, \tag{28}$$

Equation (27) reduces to

$$A''_{k-2} = [B^2_{k-1} + (-1)^{k-1}]/A_{k-1}. \tag{29}$$

Now $A''_k = yA''_{k-1} + A''_{k-2}$, and using the defining equation for y together with (25) and (27) we get from this,

$$A''_k = -[B^2_{k-1} + A_{k-1}(B_k + B_{k-2}) - (-1)^{k-1}]/A_{k-1}. \tag{30}$$

Recall now the definition of M and we get

$$A''_k + MA''_{k-1} = -[B^2_{k-1} + A_{k-1}(B_k + B_{k-2}) - A_{k-2}B_{k-1} - (-1)^{k-1} - B^2_{k-1}]/A_{k-1},$$

which reduces again, using (28), to

$$A''_k + MA''_{k-1} = -B_k, \tag{31}$$

which agrees with formula (21) when $r=1$.

We now show that if formulae (20) and (21) are true for all values of r up to and including $r=n$ they must hold for $r=n+1$. We have, using again the formulae given in Perron, page 14, $A_{(n+1)k-1} = A_{k-1}A_{nk} + B_{k-1}A_{nk-1}$, and using formulae (20) and (21), which are assumed to hold for $r=n$, this may be written: $A''_{(n+1)k-1} = (-1)^n A_{k-1}(B_{nk} + MB_{nk-1}) + (-1)^{n-1}A_{k-2}B_{nk-1}$. Putting in the value of M we get from this,

$$A''_{(n+1)k-1} = (-1)^n(A_{k-1}B_{nk} + B_{k-1}B_{nk-1}) = (-1)^n B_{(n+1)k-1},$$

which is formula (20) when $r=n+1$.

Starting now with the equation which comes from the way the continued fraction is defined,

$$B_{(n+1)k} = (n+1)bB_{(n+1)k-1} + B_{(n+1)k-2}, \tag{32}$$

we write it, using the fundamental recursion formulae (Perron, p. 14),

$$B_{(n+1)k} = (n+1)bB_{(n+1)k-1} + A_{k-2}B_{nk} + B_{k-2}B_{nk-1}.$$

This may again be written (since $B_{(n+1)k-1} = A_{k-1}B_{nk} + B_{k-1}B_{nk-1}$),

$$B_{(n+1)k} = [(n+1)b + M]B_{(n+1)k-1} - B_{k-1}(B_{nk} + MB_{nk-1}) + B_{k-2}B_{nk-1}, \tag{33}$$

but for $r=n$ we have, by hypothesis, using (22) and (20),

$$B_{nk} + MB_{nk-1} = A''_{nk}(-1)^n, \quad B_{nk-1} = (-1)^{n-1}A''_{nk-1}.$$

Also, by (7) and (9), $B_{k-1} = A'_{k-2}$, $B_{k-2} = B'_{k-2}$. Putting these values in (33) we get

$$B_{(n+1)k} = [(n+1)b + M]B_{(n+1)k-1} + (-1)^{n+1}(A'_{k-2}A''_{nk} + B'_{k-2}A''_{nk-1}), \tag{34}$$

but again, from the construction of the fraction $A''_{(n+1)k-2} = A'_{k-1}A''_{nk} + B'_{k-2}A''_{nk-1}$, and (34) becomes

$$B_{(n+1)k} = [(n+1)b + M]B_{(n+1)k-1} + (-1)^{n+1}A''_{(n+1)k-2}. \tag{35}$$

Now we have already shown that formula (20) holds for $r=n+1$ so that we can write $B_{(n+1)k-1} = (-1)^n A''_{(n+1)k-1}$ in (35), and get

$$B_{(n+1)k} = (-1)^{n+1}[((n+1)b + M)A''_{(n+1)k-1} + A''_{(n+1)k-2}]. \tag{36}$$

But again, from the succession of partial quotients of the continued fraction, $A''_{(n+1)k}=-[(n+1)b+2M]A''_{(n+1)k-1}+A''_{(n+1)k-2}$, $(n=1,2,3,\ldots)$. This in (36) gives $B_{(n+1)k}=(-1)^{n+1}(A''_{(n+1)k}+MA''_{(n+1)k-1})$, which is formula (21) when $r=n+1$. These formulae therefore hold in all cases.

From the definition of M it would seem that k must not be less than 3. With the usual conventions, however, that $A_0=1$, $B_0=0$, $A_{-1}=0$ and $B_{-1}=1$, the theorems derived above will apply when $k=1$ and $k=2$.

III.

We consider now the continued fractions (1), (2) and (17) with respect to any modulus n, and we assume that n is prime to $2b$ and to A_{k-1} and to B_{k-1}. The cases where these restrictions are not applied will be considered later. It is then possible to find two values of μ, one odd and the other even, both less than $2n$, which will satisfy the congruence

$$b\mu+2M\equiv 0 \pmod{n}. \tag{37}$$

Such a solution will furnish a zero partial quotient in the continued fraction (2) of rank μk, and the partial quotient of the same rank in fraction (1) will have the value $-2M$. Moreover, it is seen that the partial quotients of (2) read backward from this partial quotient exactly as the partial quotients of (1) read forward, so that, taken modulo n, the two fractions are inverse as far as this partial quotient. We consider first the even solution of (37), which we denote by $2m$. By the properties of inverse fractions we have (see equations (6), (7), (8) and (9)) $A_{2mk-1}\equiv A'_{2mk-1} \pmod{n}$. But by equation (4), $A_{2mk-1}=-A'_{2mk-1}$, so that

$$A_{2mk-1}\equiv 0 \pmod{n}. \tag{38}$$

IV.

The partial quotient of rank $2m$ is, as we noted above, congruent to $-2M$, the two preceding ones being a_{k-1} and a_{k-2}. The recursion formulae for a continued fraction give

$$A_{2mk}=-2MA_{2mk-1}+A_{2mk-2}, \tag{39}$$

$$A_{2mk-1}=a_{k-1}A_{2mk-2}+A_{2mk-3}, \tag{40}$$

$$A_{2mk-2}=a_{k-2}A_{2mk-3}+A_{2mk-4}. \tag{41}$$

From these we derive, using (38) the congruences,

$$\left.\begin{array}{l} A_{2mk-2}\equiv A_{2mk}\equiv A'_0A_{2mk}, \\ A_{2mk-3}\equiv -a_{k-1}A_{2mk}\equiv -A'_1A_{2mk}, \\ A_{2mk-4}\equiv (a_{k-1}a_{k-2}+1)A_{2mk}\equiv A'_2A_{2mk}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array}\right\} \pmod{n}.$$

We infer the general law

$$A_{2mk-r} \equiv (-1)^r A'_{r-2} A_{2mk} \pmod{n}. \tag{42}$$

The proof is made by complete induction. We see that the law holds for $r=1$. Suppose it true for all values of r up to and including $r=t$. Then since the continued fractions (1) and (2) are inverse modulo n as far as the partial quotient of order $2mk$ the partial quotient opposite A_{2mk-t} is the same, modulo n, as that opposite A'_t. But we have $A_{2mk-(t+1)} = A_{2mk-(t-1)} - q_{t-1} A_{2mk-t}$, where q_{t-1} is the partial quotient opposite $A_{2mk-(t-1)}$. By means of (42), which by hypothesis holds for $r=t$ this last equation may be written:

$$\begin{aligned} A_{2mk-(t+1)} &\equiv A_{2mk}[A'_{t-3}(-1)^{t-1} - q_{t-1} A'_{t-2}(-1)^t] \\ &\equiv (-1)^{t-1} A_{2mk}(A'_{t-3} + q_{t-1} A'_{t-2}) \equiv (-1)^{t-1} A_{2mk} A'_{t-1}, \end{aligned}$$

which is formula (42) for $r=t+1$. The formula then holds for all values of r. It may also be written

$$A_{2mk-(r+2)} \equiv (-1)^r A_{2mk} A'_r \pmod{n}. \tag{43}$$

Returning with this result to equation (4) we obtain the congruence

$$\begin{aligned} A_{2mk-(kr+1)} &\equiv (-1)^{kr+1} A_{2mk}(-1)^{r-1} A_{kr-1} \pmod{n} \\ &\equiv (-1)^{r(k+1)} A_{2mk} A_{kr-1} \pmod{n}. \end{aligned} \tag{44}$$

Put $r=m$ in this congruence and get $A_{mk-1} \equiv (-1)^{m(k+1)} A_{2mk} A_{mk-1} \pmod{n}$, or

$$A_{mk-1}[A_{2mk} - (-1)^{m(k+1)}] \equiv 0 \pmod{n}. \tag{45}$$

In the same way, starting with equation (5), we get the congruence

$$(-1)^{r(k-1)} A_{2mk} A_{rk} \equiv A_{2mk-rk-2} - M A_{2mk-rk-1} \pmod{n},$$

and putting $r=m$ in this we get $(-1)^{m(k-1)} A_{2mk} A_{mk} \equiv A_{mk-2} - M A_{mk-1}$. But $A_{mk} = mb A_{mk-1} + A_{mk-2}$, whence $(-1)^{m(k-1)} A_{2mk} A_{mk} \equiv A_{mk} - (mb+M) A_{mk-1}$, which we may write in the form

$$A_{mk}[(-1)^{m(k-1)} - A_{2mk}] \equiv (mb+M) A_{mk-1} \pmod{n}. \tag{46}$$

Now by (45) either A_{mk-1} is congruent to zero, or else $A_{2mk} - (-1)^{m(k-1)}$ is congruent to zero, or perhaps both factors are. But if A_{mk-1} is congruent to zero then A_{mk} is not, since, by the equation $A_{mk} B_{mk-1} - A_{mk-1} B_{mk} = (-1)^{mk}$, A_{mk} and A_{mk-1} can have no common factor. Therefore by (46), if A_{mk-1} is congruent to zero, so is $A_{2mk} - (-1)^{m(k-1)}$ also. If, on the other hand, A_{mk-1} is not congruent to zero, then by (45) $A_{2mk} - (-1)^{m(k-1)}$ must be. Therefore we have in all cases,

$$A_{2mk} \equiv (-1)^{m(k-1)} \pmod{n}. \tag{47}$$

V.

Returning to (42) with this last result, we get

$$A_{2mk-r} \equiv (-1)^{r+m(k+1)} A'_{r-2} \pmod{n}. \tag{48}$$

VI.

From the equation $A_{2mk}B_{2mk-1}-A_{2mk-1}B_{2mk}=1$ we have, using (38) and (47),

$$B_{2mk-1}\equiv(-1)^{m(k-1)} \pmod{n}. \tag{49}$$

VII.

Combining (47) with (44) we obtain

$$A_{2mk-rk+1}\equiv(-1)^{(k+1)(m+r)}A_{rk-1} \pmod{n}. \tag{50}$$

This formula shows that apart from sign the values of A_{rk-1} read backward and forward the same, from $r=0$ to $r=2mk-1$. When k is odd the signs are all the same, while if k is even the signs alternate. It is easily shown that the same theorems apply to the continued fraction (2). Corresponding theorems hold for the denominators B_{rk-1}, but to establish them we must consider continued fraction (17).

VIII.

It will be observed that after the partial quotient y in (17), the succession of partial quotients are the same as in (2). Let us call P_r/Q_r the r-th convergent to the fraction

$$\overline{[a_{k-1},\, a_{k-2},\, \ldots,\, a_2,\, a_1,\, -(b\mu+2M)]}, \quad (\mu=2, 3, \ldots). \tag{51}$$

We have then,

$$A''_r=A''_kP_r+A''_{k-1}Q_r, \tag{52}$$

$$B''_r=B''_kP_r+B''_{k-1}Q_r, \tag{53}$$

$$A'_r=A'_kP_r+A'_{k-1}Q_r, \tag{54}$$

$$B'_r=B'_kP_r+B'_{k-1}Q_r. \tag{55}$$

Solving (52) and (53) for P_r and Q_r we get

$$P_r=(-1)^k(B''_{k-1}A''_r-A''_{k-1}B''_r), \tag{56}$$

$$Q_r=(-1)^k(-B''_kA''_r+A''_kB''_r). \tag{57}$$

Similarly, from (54) and (55),

$$P_r=(-1)^k(B'_{k-1}A'_r-A'_{k-1}B'_r), \tag{58}$$

$$Q_r=(-1)^k(-B'_kA'_r+A'_kB'_r). \tag{59}$$

From (56) and (58) we get

$$B''_{k-1}A''_r-A''_{k-1}B''_r=B'_{k-1}A'_r-A'_{k-1}B'_r, \tag{60}$$

and from (57) and (59) we get

$$-B''_kA''_r+A''_kB''_r=-B'_kA'_r+A'_kB'_r. \tag{61}$$

Eliminate B''_r from (60) and (61) and we get

$$B'_r(A''_{k-1}A'_k-A'_{k-1}A''_k)-A'_r(A''_{k-1}B'_k-A''_kB'_{k-1})=(-1)^kA''_r. \tag{62}$$

We proceed to find the values of the quantities in the parentheses. We know from (6) that $A'_{k-1}=A_{k-1}$. Also from (30),

$$A''_k=-[B^2_{k-1}+A_{k-1}(B_k+B_{k-2})-(-1)^{k-1}]/A_{k-1};$$

while from (13) $A'_k=-(bA_{k-1}+2A_{k-2}+B_{k-1})$, and from (25) $A''_{k-1}=B_{k-1}$. Putting these values in the coefficient of B'_r in (62), replacing B_k by its value $bB_{k-1}+B_{k-2}$, and remembering that $A_{k-1}B_{k-2}-A_{k-2}B_{k-1}$ is equal to $(-1)^{k-1}$, we get easily $A''_{k-1}A'_k-A''_kA'_{k-1}=(-1)^{k-1}$. The coefficient of A'_r also reduces. For we have $B'_k=-(b+M)B'_{k-2}+B'_{k-2}$; or, using (8) and (9),

$$B'_k=-(b+M)A_{k-2}+B_{k-2}.$$

Using the same reductions as before, the coefficient of A'_r may be made to take the form $(-1)^{k-1}M$, so that (62) reduces to

$$A''_r=MA'_r-B'_r. \tag{63}$$

If now in this last equation we put $rk-1$ in place of r, and make use of (20) and (50) this may be written:

$$(-1)^rB'_{rk-1}\equiv B_{rk-1}-MA_{rk-1} \pmod{n}. \tag{64}$$

Again, put $r=2km$ in (63), and note that $A'_{2mk}\equiv(-1)^{m(k+1)} \pmod{n}$, $B_{2mk-1}\equiv(-1)^{m(k+1)} \pmod{n}$, and by (22) $A''_{2mk}=B_{2mk}+MB_{2mk-1}$, and get on reducing,

$$B_{2mk}\equiv-B'_{2mk} \pmod{n}. \tag{65}$$

IX.

The partial quotient of order $2mk$ in continued fraction (2) is zero, modulo n, so that we may write the following congruences:

$$\left.\begin{aligned}
&B'_{2mk}\equiv B'_{2mk-2}\,,\\
&B'_{2mk-1}\equiv a_1B'_{2mk-2}+B'_{2mk-3}\equiv(-1)^{m(k+1)},\\
&B'_{2mk-2}\equiv a_2B'_{2mk-3}+B'_{2mk-4}\,,\\
&\dots\dots\dots\dots\dots\dots,
\end{aligned}\right\}\pmod{n}$$

whence we obtain:

$$\left.\begin{aligned}
&B'_{2mk}\equiv B'_{2mk}\equiv-B_{2mk}, \text{ by (65)},\\
&B'_{2mk-1}\equiv(-1)^{m(k+1)},\\
&B'_{2mk-2}\equiv-B_{2mk}\,,\\
&B'_{2mk-3}\equiv-a_1(-B_{2mk})+(-1)^{m(k+1)},\\
&B'_{2mk-4}\equiv(a_1a_2+1)(-B_{2mk})-a_2(-1)^{m(k+1)},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\}\pmod{n}$$

From these congruences we infer the following which is easily established by complete induction

$$B'_{2mk-r}\equiv(-1)^r(-A_{r-2}B_{2mk}-(-1)^{m(k+1)}B_{r-2}) \pmod{n}. \tag{66}$$

Put $r=2mk-1$ in this last congruence and get

$$B'_1 \equiv A_{2mk-3}B_{2mk}+(-1)^{m(k+1)}B_{2mk-3} \pmod{n}. \tag{67}$$

But $B'_1=1$ and $A_{2mk-1}-a_{k-1}A_{2mk-2}=A_{2mk-3}$, $B_{2mk-1}-a_{k-1}B_{2mk-2}=B_{2mk-3}$, so that $A_{2mk-3}\equiv a_{k-1}(-1)^{m(k+1)} \pmod{n}$, and (67) reduces easily to

$$B_{2mk}\equiv -B_{2mk-2} \pmod{n}. \tag{68}$$

But since the partial quotient of (1) of rank $2mk$ is $-2M$, and $B_{2mk-1}\equiv(-1)^{mk+1}$, we have $B_{2mk}\equiv -2M(-1)^{mk+1}+B_{2mk-2} \pmod{n}$, so that, by (68),

$$B_{2mk}\equiv -M(-1)^{mk+1} \pmod{n}. \tag{69}$$

Returning with this value to (66) we get

$$B'_{2mk-r}\equiv(-1)^{m(k+1)+r+1}(B_{r-2}-MA_{r-2}) \pmod{n}. \tag{70}$$

X.

Reading backward from B_{2mk}, as in the derivation of (66) we arrive at the formula

$$B_{2mk-r}\equiv(-1)^{m(k+1)+r+1}(B'_{r-2}-MA'_{r-2}) \pmod{n}. \tag{71}$$

In this formula put $r=tk+1$; then

$$B_{2mk-tk-1}\equiv(-1)^{m(k+1)+tk}(B'_{tk-1}-MA'_{tk-1}) \pmod{n}, \tag{72}$$

and this with (63) gives $B_{2mk-tk-1}\equiv(-1)^{m(k+1)+tk-1}A''_{tk-1} \pmod{n}$. But, by (20) this gives

$$B_{2mk-tk-1}\equiv(-1)^{(m+t)(k+1)}B_{tk-1} \pmod{n}, \tag{73}$$

which is the same relation between the B's as (50) between the A's.

XI.

From (72) and (73) we get easily

$$B_{tk-1}\equiv(-1)^{t}(B'_{tk-1}-MA'_{tk-1}) \pmod{n}, \tag{74}$$

while (71) gives the corresponding formula:

$$B'_{tk-1}\equiv(-1)^{t}(B_{tk-1}-MA_{tk-1}) \pmod{n}. \tag{75}$$

XII.

Let m' now be the odd value of μ less than $2n$ which satisfies the congruence, $\mu b+2M\equiv 0 \pmod{n}$, and suppose first $2m$ is greater than this odd value so that $2m-m'=n$. We have

$$A_{2mk-1}\equiv 0\equiv A_{nk}A_{m'k-1}+A_{nk-1}B_{m'k-1} \pmod{n}, \tag{76}$$

$$A_{2mk}\equiv(-1)^{m(k+1)}\equiv A_{nk}A_{m'k}+A_{nk-1}B_{m'k} \pmod{n}. \tag{77}$$

Eliminate A_{nk-1} from these two equations and get

$$B_{m'k-1}\equiv A_{nk}(-1)^{k(m-m')+m} \pmod{n}. \tag{78}$$

But by (50) and (73) we know that

$$A_{nk-1}\equiv(-1)^{(k+1)(m'+m)}A_{m'k-1} \pmod n, \tag{79}$$

$$B_{nk-1}\equiv(-1)^{(k+1)(m'+m)}B_{m'k-1} \pmod n, \tag{80}$$

so that

$$A_{nk}\equiv -B_{nk-1} \pmod n. \tag{81}$$

But $A_{2nk-1}=A_{nk}A_{nk-1}+B_{nk-1}A_{nk-1}=A_{nk-1}(A_{nk}+B_{nk-1})$, so that by (81) we have

$$A_{2nk-1}\equiv 0 \pmod n. \tag{82}$$

Also $B_{2nk}=A_{nk}B_{nk}+B_{nk}B_{nk-1}$; and this by (81) gives

$$B_{2nk}\equiv 0 \pmod n. \tag{83}$$

Again, $A_{2nk}=A_{nk}A_{nk}+A_{nk-1}B_{nk}\equiv -A_{nk}B_{nk-1}+A_{nk-1}B_{nk}$, by (81), and by (82) and (83) this gives

$$A_{2nk}\equiv(-1)^{nk-1} \pmod n. \tag{84}$$

And, finally, from the equation $A_{2nk}B_{2nk-1}-A_{2nk-1}B_{2nk}=1$, we derive

$$B_{2nk-1}\equiv(-1)^{nk-1} \pmod n. \tag{85}$$

XIII.

Suppose next that m' is greater than $2m$ so that $m'-2m=n$. We have then

$$A_{m'k}\equiv A_{nk}A_{2mk}+A_{nk-1}B_{2mk}, \quad A_{m'k-1}=A_{nk}A_{2mk-1}+A_{nk-1}B_{2mk-1}.$$

Put in these the values of A_{2mk}, etc., already derived, and these equations become

$$(-1)^{mk+1}A_{m'k} \equiv A_{nk}-MA_{nk-1} \pmod n. \tag{86}$$

$$(-1)^{mk+1}A_{m'k-1}\equiv A_{nk-1} \pmod n. \tag{87}$$

$$(-1)^{mk+1}B_{m'k} \equiv B_{nk}-MB_{nk-1} \pmod n. \tag{88}$$

$$(-1)^{mk+1}B_{m'k-1}\equiv B_{nk-1} \pmod n. \tag{89}$$

From these we get

$$A_{nk}+B_{nk-1}\equiv(-1)^{mk+1}(A_{mk}+MA_{mk-1}+B_{mk-1}). \tag{90}$$

But, recalling the definition of m' we observe that the fractions (1) and (2), as far as the km'—1-st term, are inverse modulo n, so that

$$A_{m'k-1}\equiv A'_{m'k-1} \pmod n. \tag{91}$$

$$A_{m'k-2}\equiv B'_{m'k-1} \pmod n. \tag{92}$$

$$B_{m'k-1}\equiv A'_{m'k-2} \pmod n. \tag{93}$$

$$B_{m'k-2}\equiv B'_{m'k-2} \pmod n. \tag{94}$$

Also by (5), since m' is odd, $-A_{m'k}=A_{m'k}+MA_{m'k-1}$.

Further, since the $m'k$-th partial quotient is zero in (2),

$$A_{mk}\equiv A_{mk-2} \pmod n, \tag{95}$$

whence, from (92) and (95)

$$A_{mk}+MA_{mk-1}\equiv -B_{mk-1} \pmod n, \tag{96}$$

which in (90) gives $A_{nk}+B_{nk-1}\equiv 0 \pmod n$, and this is the same formula as (81) for the case where $2m$ is greater than m'. Formulae (82), (83), (84) and (85) therefore hold whether $2m$ is greater or less than m'.

XIV.

From the equations

$$A_{4nk} = A_{2nk}+A_{2nk-1}B_{2nk}, \tag{97}$$

$$A_{4nk-1}=A_{2nk}A_{2nk-1}+A_{2nk-1}B_{2nk}, \tag{98}$$

$$B_{4nk} = B_{2nk}A_{2nk}+B_{2nk-1}B_{2nk}, \tag{99}$$

$$B_{4nk-1}=B_{2nk}A_{2nk-1}+B_{2nk-1}B_{2nk-1}, \tag{100}$$

we now derive

$$A_{4nk} \equiv B_{4nk-1}\equiv 1 \pmod n, \tag{101}$$

$$A_{4nk-1}\equiv B_{4nk} \equiv 0 \pmod n. \tag{102}$$

From these results it appears that taken modulo n the series of convergents repeat themselves with a period of $4nk$, but if k is odd as well as n, the period is $2nk$.

XV.

We now extend the above results to the fraction

$$\overline{(a_1, a_2, a_3, \ldots, a_{k-1}, \mu b+c)}, \qquad (\mu=1, 2, 3, \ldots), \tag{103}$$

where, as before, n is prime to $2b$, and to A_{k-1} and B_{k-1}. It is clear that there will be a partial quotient $\mu b+c\equiv 0 \pmod n$, after which the fraction is of the type (1). Call P_r/Q_r the r-th convergent of (103) and as before A_r/B_r the r-th convergent of (1). Then we have, μ being determined by the congruence $\mu b+c\equiv 0 \pmod n$,

$$P_{\mu+2nk-1}\equiv P_{\mu k}A_{2nk-1}+P_{\mu k-1}B_{2nk-1} \pmod n. \tag{104}$$

$$P_{\mu+2nk} \equiv P_{\mu k}A_{2nk}+P_{\mu k-1}B_{2nk} \pmod n. \tag{105}$$

$$Q_{\mu+2nk-1}\equiv Q_{\mu k}A_{2nk-1}Q_{\mu k-1}B_{2nk-1} \pmod n. \tag{106}$$

$$Q_{\mu+2nk} \equiv Q_{\mu k}A_{2nk}+Q_{\mu k-1}B_{2nk} \pmod n. \tag{107}$$

But we have also,

$$P_{\mu+2nk-1}\equiv P_{2nk}P_{\mu k-1}+P_{2nk-1}Q_{\mu k-1} \pmod n. \tag{108}$$

$$P_{\mu+2nk} \equiv P_{2nk}P_{\mu k}+P_{2nk-1}Q_{\mu k} \pmod n. \tag{109}$$

$$Q_{\mu+2nk-1}\equiv Q_{2nk}P_{\mu k-1}+Q_{2nk-1}Q_{\mu k-1} \pmod n. \tag{110}$$

$$Q_{\mu+2nk} \equiv Q_{2nk}P_{\mu k}+Q_{2nk-1}Q_{\mu k} \pmod n. \tag{111}$$

Using (82), (83), (84) and (85), we derive from these,

$$P_{\mu k-1}[(-1)^{nk-1}-P_{2nk}]\equiv P_{2nk-1}Q_{\mu k-1} \pmod n. \tag{112}$$

$$P_{\mu k}[(-1)^{nk-1}-P_{2nk}] \equiv P_{2nk-1}Q_{\mu k} \pmod n. \tag{113}$$

$$Q_{\mu k-1}[(-1)^{nk-1}-Q_{2nk-1}]\equiv Q_{2nk}P_{\mu k-1} \pmod n. \tag{114}$$

$$Q_{\mu k}[(-1)^{nk-1}Q_{2nk-1}]\equiv Q_{2nk}P_{\mu k} \pmod n. \tag{115}$$

Eliminate now P_{2nk-1} from (112) and (113) and obtain

$$(P_{\mu k-1}Q_{\mu k}-P_{\mu k}P_{\mu k-1})\,[(-1)^{nk-1}-P_{2nk}]\equiv 0 \pmod{n}.$$

Then, since the first factor on the left is ± 1, we have

$$P_{2nk}\equiv(-1)^{nk-1} \pmod{n}. \tag{116}$$

Similarly, from (114) and (115) we get,

$$Q_{2nk-1}\equiv(-1)^{nk-1} \pmod{n}. \tag{117}$$

Also, by (112) and (113), $P_{2nk-1}Q_{\mu k-1}\equiv 0 \pmod{n}$, $P_{2nk-1}Q_{\mu k}\equiv 0 \pmod{n}$, and since $Q_{\mu k}$ and $Q_{\mu k-1}$ can not both be congruent to zero on account of the equation $P_{\mu k}Q_{\mu k-1}-P_{\mu k-1}Q_{\mu k}=(-1)^{\mu k}$, we must have

$$P_{2nk-1}\equiv 0 \pmod{n}, \tag{118}$$

and, similarly,

$$Q_{2nk}\equiv 0 \pmod{n}. \tag{119}$$

XVI.

The above results may be extended to fractions which have a set of "irregular" or non-periodic partial quotients followed by partial quotients of the sort considered in fraction (103). Such a fraction would be of the form

$$(q_1,\,q_2,\,\ldots,\,q_r,\,\overline{a_1,\,a_2,\,\ldots,\,a_{k-1},\,\mu b+c}), \qquad (\mu=1,\,2,\,3,\,\ldots).$$

For this fraction there will be r non-periodic convergents, after which the periodicity begins, the length of the period being the same as for the fraction (103). The two successive convergents which close each period will, however, not be congruent respectively to ± 1, 0 and 0, ± 1, but to the $(r-1)$-st and the r-th convergents respectively of the fraction $(q_1,\,q_2,\,\ldots,\,q_r)$.

XVII.

We consider now the special cases which have been ruled out in the statement of our theorems, and take up first the case where b is congruent to zero, modulo n. This will include also the case where b is actually zero, in which case the fraction is an ordinary purely periodic continued fraction. We may write it in the form $(\overline{a_1,\,a_2,\,a_3,\,\ldots,\,a_k})$. The convergents of order k and $k-1$ are connected with those of order $2k$ and $2k-1$ by the following equations:

$$\begin{aligned} A_{2k}\ &=A_k^2+A_{k-1}B_k, & B_{2k}\ &=B_kA_k+B_{k-1}B_k,\\ A_{2k-1}&=A_kA_{k-1}+A_{k-1}B_{k-1}, & B_{2k-1}&=B_kA_{k-1}+B_{k-1}^2. \end{aligned}$$

Now, these are seen to result from the same process as that by which two linear homogeneous substitutions are compounded, so that if we call T_k the substitution,

$$T_k=\begin{pmatrix} A_k, & B_k\\ A_{k-1}, & B_{k-1}\end{pmatrix},$$